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Numerical Solutions of Applications in Ordinary Differential Equation by Implicit Multistep Method

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Abstract: In this research, an implicit multistep method which is Adams Moulton three step method is used for solving application of first and second order ordinary differential equations (ODEs). The applications in first order ODEs which is decay will be solved using Adams Moulton three step method directly and for the application of second order ODEs which is electric circuit, at first it will be reduced to the system of first order ODEs, then will be solved using the Adams Moulton three step method. The numerical solution that has been obtained using Adams Moulton three step method is being compared with the exact solution to determine the accuracy of the Adams Moulton three step method towards the exact solution. The absolute error is discussed further in this research. The results for the Adams Moulton three step method were obtained using MATLAB 2021. It was observed that the Adams Moulton three step method were efficient in solving application of first and second order ODEs.

Keywords: Adams Moulton Three Step Method, Ordinary Differential Equation, Multistep Method

1. Introduction

A differential equation (DE) is an equation in which variables and their rates of change are involved. Ordinary differential equations (ODEs) are defined as a function equation with only one independent variable. The order of the DEs can be determined as the order of the highest derivatives occur in the equations. In this study, we will focus on first order and second order ODEs.

The general equation for first order ODEs is considered as

$$y' = f(x, y) \qquad \text{Eq. 1}$$

and the general equation for second order ODEs is

$$y'' = f(x, y, y')$$ Eq. 2

The linear multistep method (LMM) is one of the most common methods for solving ODEs numerically. The Adams Moulton method is an example of LMM. LMM can be determined using numerical integration, Taylor series, or the Lagrange interpolation polynomial. The general LMM is written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$ Eq. 3

where α$_j$ and β$_j$ are constants, $\alpha_k \neq 0$ and at least one of the coefficients α_0 and β_0 is non-zero. If $\beta_k = 0$, the Eq. 3 is called explicit, and if $\beta_k \neq 0$, it is called implicit. For the implicit method to be successful, it is important to have a good first iteration. We may use an explicit multistep method to get a successful initial iteration. Therefore, in this study corrector methods is implicit multistep method, whereas predictor method is explicit multistep methods.

[1] proposed linear multistep method (LMM) for second order initial value problem (IVP) solutions and discovered that the order-seven method was consistent, zero-stable, and convergent with a suitable region of absolute stability. Using the collocation technique, the authors used the power series function to get approximate solutions. The method solved second-order IVPs without having to reduce the ODEs to a first-order system.

[2] studied the sunspot as an autonomous dynamical system of model for the growth and decay stages of sunspots. The model reproduces both the sunspot growth and decay phases. [3] studied about the Gompertz and Verhulst frameworks for growth and decay description Marcel. Numerous examples demonstrate that the Verhulst and Gompertz models are overly reductive (or restrictive) in their original intent. The goal of this note is to condense the growth and decay features into one simple differential equation.

The fractional order RC electric circuit in the time and frequency domain was investigated by [4]. In the Caputo sense, they apply the Laplace transform of the fractional derivative. The delay, rise, and settling times are of relevance in the time domain, whereas the cutoff frequency, bandwidth, and asymptotes in low and high frequencies are of particular importance in the frequency domain. The order of the differential equation affects all these values.

[5] investigates the use of equivalent circuits to solve ordinary and partial differential equations numerically. With the help of equivalent electrical networks, numerical methods are developed to solve certain types of linear and nonlinear partial differential equations to any desired degree of accuracy.

The objectives of this study are to examine the applications of first order and second order ODEs, to solve the problem of applications of first order and second order ODEs using implicit Adams Moulton method, and to compare the result with the exact solutions.

2. Methodology

This paper will be discussed on solving the application problems of first and second order ODEs. The decay problem is chosen as an application of first order ODEs while problem involving electrical circuit will be considered as an application of second order ODEs.

2.1 Applications of ordinary differential equations
2.1.1 Decay

The independent variable will be denoted by t because the application in this section deal with functions of time. N' will signify the derivative of N with respect to t if N is a function of t; hence

$$N' = \frac{dN}{dt} \qquad \text{Eq. 4}$$

The exponential model, in which the rate of change of a quantity N is proportional to N, is one of the most common mathematical models for a physical process. Thus the initial value problem

$$N' = aN, \qquad N(t_0) = N, \qquad \text{Eq. 5}$$

Where N satisfies this equation decays exponentially if $a < 0$.

2.1.2 Electric circuit

The voltage drop across an electromotive force is stated as $-E(t)$. As a result of Kirchhoff's loop law, we have

$$RI + L\frac{dI}{dt} + \frac{1}{C}q - E(t) = 0 \qquad \text{Eq. 6}$$

The relationship between q and I is

$$I = \frac{dq}{dt}, \qquad \frac{dI}{dt} = \frac{d^2q}{dt^2} \qquad \text{Eq. 7}$$

By substituting Eq. 7 into Eq. 6 we obtain

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t) \qquad \text{Eq. 8}$$

The initial conditions for q are

$$q(0) = q_0, \qquad \left.\frac{dq}{dt}\right|_{t=0} = I(0) = I_0 \qquad \text{Eq. 9}$$

We differentiate Eq. 6 with respect to t and then directly substitute Eq. 7 into the resulting equation to obtain a differential equation for the current, which is

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{E(t)}{dt} \qquad \text{Eq. 10}$$

$I(0) = I_0$ is the first initial condition. The second initial condition is obtained from Eq. 6 by solving $\frac{dI}{dt}$ and then setting $t = 0$. Thus,

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I_0 - \frac{1}{LC}q_0 \qquad \text{Eq. 11}$$

An expression for the current can be gotten either by solving Eq. 11 directly or by solving Eq. 8 for the charge and then differentiating that expression.

2.2 Numerical method

Adams Moulton three step method will be used to solve the ODEs. For this method, Adam Bashforth four step method is considered as a predictor.

The four step Adams Bashforth method need four initial values to start the calculation, so it needs to used other methods to get these initial values. In this research, Runge-Kutta fourth order is used to find the value for $i = 1$ until $i = 4$. The formula for Runge-Kutta fourth order is

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$K_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$K_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad \text{Eq. 12}$$

Then the calculation will be continued by finding the predictor values using Adams Bashforth four step method. The formula of Adams Bashforth four step method is

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) \\ + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] \qquad \text{Eq. 13}$$

The predictor values will be used to compute the solutions using Adams Moulton three step method is

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) \\ - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})] \qquad \text{Eq. 14}$$

3. Result and Discussion

For this research, the Adams Moulton will be focus on solving the application of first and second order of ordinary differential equations (ODEs). The problem that will be solved are taken from the books by [6] to obtain the solutions. Next, the solutions obtained from the Adams Moulton method will be compared with the exact solution.

3.1 Test problem 1 (Decay)

A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 milligrams of the material present and after two hours, it is observed that the material has lost 10 percent of its original mass, then the constant $a = -0.053 < 0$. Determine the amount of radioactive material in interval of time, $0 \leq t \leq 5$ hours.
Exact solution:

$$N = 50e^{-0.053t} \qquad \text{Eq. 15}$$

Source: (Bronson et al., 2006) [6]

Let N is the radioactive material and t is time in hours, the equation for decay problem is given by

$$\frac{dN}{dt} = -0.053N \qquad \text{Eq. 16}$$

with the initial condition $N(0) = 50$ and interval time, $0 \leq t \leq 5$. For this problem, we consider the step size $h = 0.5$ and $h = 0.05$.

Table 1: Numerical result of Three Step Adams Moulton method for solving Test problem 1 with $h = 0.5$

t_i	Exact solution	Three step Adams Moulton Method	
		Numerical Solution	Absolute error
	Y	y	$|Y - y|$
0.0	50.00000	50.00000	0.0000000e+00
0.5	48.69240	48.69240	5.4213000e-09
1.0	47.41900	47.41900	1.0559000e-08
1.5	46.17890	46.17890	1.5424400e-08
2.0	44.97123	44.97123	3.1949600e-08
2.5	43.79515	43.79515	2.0754300e-08
3.0	42.64982	42.64982	3.7412600e-08
3.5	41.53444	41.53444	5.3187500e-08
4.0	40.44823	40.44823	6.8111800e-08
4.5	39.39043	39.39043	8.2219000e-08
5.0	38.36030	38.36030	9.5541900e-08

Table 2: Numerical result of Three Step Adams Moulton method for solving Test problem 1 with $h = 0.05$

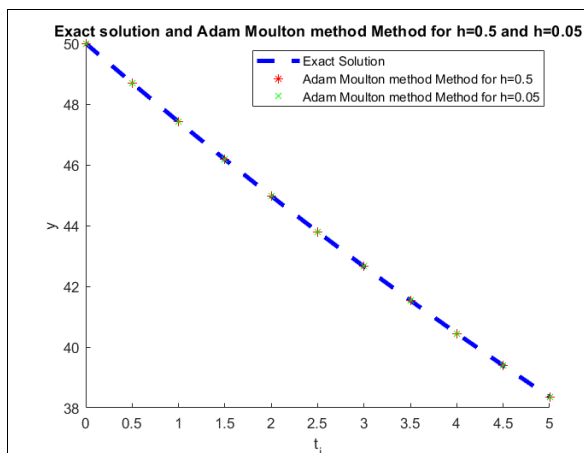| t_i	Exact solution	Three step Adams Moulton Method	
		Numerical Solution	Absolute error
	Y	y	$|Y - y|$
0.0	50.00000	50.00000	0.00000e+00
0.5	48.69240	48.69240	1.03739e-12
1.0	47.41900	47.41900	2.67164e-12
1.5	46.17890	46.17890	4.21352e-12
2.0	44.97123	44.97123	5.68434e-12
2.5	43.79515	43.79515	7.06990e-12
3.0	42.64982	42.64982	8.37019e-12
3.5	41.53444	41.53444	9.60654e-12
4.0	40.44823	40.44823	1.07789e-11
4.5	39.39043	39.39043	1.18803e-11
5.0	38.36030	38.36030	1.29177e-11

Figure 1: Graph for Exact solution and Adams Moulton method for $h = 0.5$ and $h = 0.05$
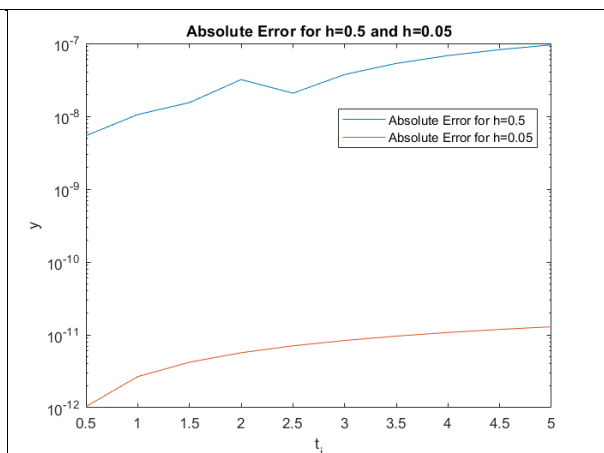
Figure 2: Graph for Absolute Error for step size 0.5 and 0.05

3.2 Test Problem 2 (Electric circuit)

An *RCL* circuit connected in series has $R = 180$ ohms, $C = \frac{1}{280}$ farad, $L = 20$ Henries, and an applied voltage $E(t) = 10\sin t$. Assume no initial charge on the capacitor, but an initial current of 1 ampere at $t = 0$ when the voltage is first applied. Find the subsequent charge on the capacitor in interval of time $0 \le t \le 1$.
Exact solution:

$$x(t) = e^{\frac{1}{2}t\sqrt{5}-\frac{9}{2}t}\left(\frac{37}{180}\sqrt{5} + \frac{1}{180}\right) \qquad \text{Eq. 17}$$
$$+ e^{-\frac{1}{2}t\sqrt{5}-\frac{9}{2}t}\left(-\frac{37}{180}\sqrt{5} + \frac{1}{180}\right)$$
$$- \frac{1}{90}\cos(t) + \frac{1}{45}\sin(t)$$

$$y(t) = \frac{1}{2}\left(-9 + \sqrt{5}\right)e^{\frac{1}{2}(-9+\sqrt{5})t}\left(-\frac{37}{180}\sqrt{5} + \frac{1}{180}\right) \qquad \text{Eq. 18}$$
$$+ \frac{1}{2}\left(-9 + \sqrt{5}\right)e^{\frac{1}{2}(9+\sqrt{5})t}\left(-\frac{37}{180}\sqrt{5}\right.$$
$$\left. + \frac{1}{180}\right) + \frac{1}{90}\sin(t) + \frac{1}{45}\cos(t)$$

Source: (Bronson et al., 2006) [6]
Substituting the given quantities into Eq. 8, we obtain

$$q'' + 9q'' + 19q = \frac{1}{2}sint \qquad \text{Eq. 19}$$

the initial conditions $q(0) = 0$ and $q(0) = 1$.
Reduce Eq. 22 to the system of first order ODEs yields

$$q_1 = q_2 \qquad \text{Eq. 20}$$

$$q_2 = -9q_2 - 19q_1 + \frac{1}{2}sint \qquad \text{Eq. 21}$$

We will solve Eq. 20 and Eq. 21 in interval $0 \le t \le 1$ with step sizes $h = 0.1$ and $h = 0.01$.

Table 3: Numerical result of Three Step Adams Moulton method for solving Test problem 2 with $h = 0.1$

t_i	Exact solution		Adams Moulton			
			Numerical Solution		Absolute error	
	Y_1	Y_2	y_1	y_2	$\lvert Y_1 - y_1 \rvert$	$\lvert Y_2 - y_2 \rvert$
0.0	0.00000	1.00000	0.000000	1.000000	0.000000e+00	0.000000e+00
0.1	0.06396	0.35595	0.063790	0.356980	1.772310e-04	1.032080e-03
0.2	0.08243	0.05342	0.082230	0.054580	1.975110e-04	1.161490e-03
0.3	0.08044	-0.07299	0.080270	0.072020	1.639650e-04	9.765800e-04
0.4	0.07069	-0.11230	0.070570	-0.111570	1.198890e-04	7.261450e-04
0.5	0.05929	-0.11145	0.059640	-0.113460	3.449750e-04	2.012250e-03
0.6	0.04892	-0.09452	0.049360	-0.097130	4.411810e-04	2.609940e-03
0.7	0.04052	-0.07332	0.040830	-0.075220	3.125690e-04	1.900600e-03
0.8	0.03422	-0.05319	0.034450	-0.054650	2.335490e-04	1.454260e-03
0.9	0.02978	-0.03621	0.029950	-0.037300	1.708540e-04	1.089220e-03
1.0	0.02685	-0.02284	0.026960	-0.023540	1.046160e-04	6.996710e-04

Table 4: Numerical result of Three Step Adams Moulton method for solving test problem 2 with $h = 0.01$

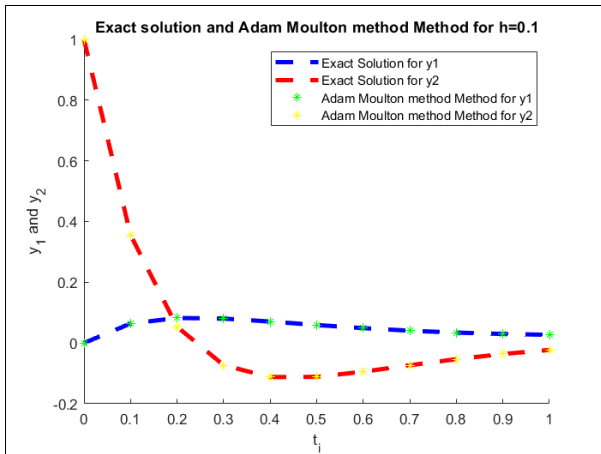| t_i	Exact solution		Adams Moulton			
			Numerical Solution		Absolute error	
	Y_1	Y_2	y_1	y_2	$\|Y_1 - y_1\|$	$\|Y_2 - y_2\|$
0.0	0.00000	1.00000	0.000000	1.000000	0.000000e+00	0.000000e+00
0.1	0.06396	0.35595	0.063960	0.355950	2.426060e-08	1.415050e-07
0.2	0.08243	0.05342	0.082430	0.053420	4.018590e-08	2.370040e-07
0.3	0.08044	-0.07299	0.080440	-0.072990	3.693000e-08	2.208520e-07
0.4	0.07069	-0.11230	0.070690	-0.112300	2.826080e-08	1.721330e-07
0.5	0.05929	-0.11145	0.059290	-0.111450	1.961060e-08	1.224270e-07
0.6	0.04892	-0.09452	0.048920	-0.094520	1.270510e-08	8.205840e-08
0.7	0.04052	-0.07332	0.040520	-0.073320	7.752820e-09	5.255580e-08
0.8	0.03422	-0.05319	0.034220	-0.053190	4.436140e-09	3.232620e-08
0.9	0.02978	-0.03621	0.029780	-0.036210	2.330170e-09	1.907270e-08
1.0	0.02685	-0.02284	0.026850	-0.022840	1.057690e-09	1.070760e-08

Figure 3: Graph for Exact solution and Adams Moulton method for $h = 0.1$
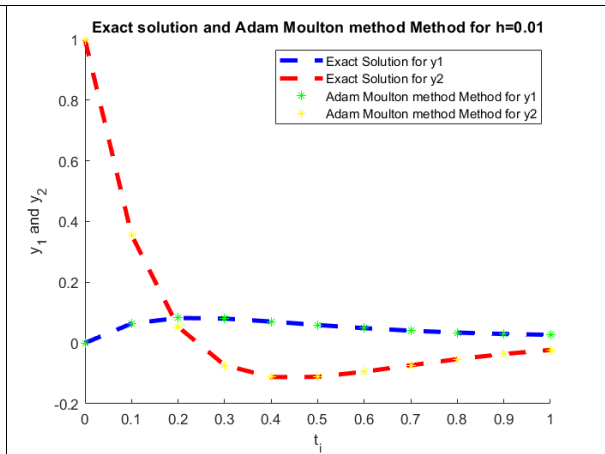
Figure 4: Graph for Exact solution and Adams Moulton method for $h = 0.01$
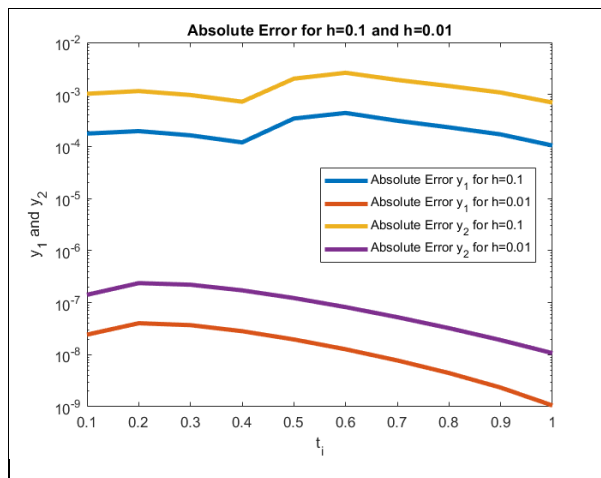

Figure 5: Graph for Absolute Error for step size 0.1 and 0.01

The results of the test problems are solved using the Adams Moulton three step method. For each test problem, the result will be compared in term of the absolute error to identify the best result. The absolute error also presents graphically by MATLAB software.

For Test problem 1, the application of first order ODEs which is decay is solve directly by using Adams Moulton three step method. The solution for Test problem 1 is shown in Table 1 and Table 2. From the Figure 2 we can see the difference in absolute error for step size 0.5 and 0.05. Table 2 has the smaller absolute errors compare to Table 1

For Test problem 2, the application of second order ODEs which is electric circuit is solve by using Adams Moulton three step method after the equation is reduce to system of first order ODEs. The solution for Test problem 2 is shown in Table 3 and Table 4. From the Figure 5 we can see the difference in absolute error for step size 0.1 and 0.01. Table 4 has the smaller absolute errors compare to Table 3.

4. Conclusion

This research emphasizes solving application of first and second order ODEs using the three step Adams Moulton method. The second order test problem must be reduced to system of first order ODEs. The result is calculated by using MATLAB software with difference step sizes for each problem. The results show that the smaller the step size used, the better the result.

Acknowledgement

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