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Fourth-Order Runge-Kutta Method for Solving Applications of System of First-Order Ordinary Differential Equations

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Abstract: This paper study the applicability and the efficiency of the fourth-order Runge-Kutta method (RK4), resulting in solving the application of system of first-order ordinary differential equations (ODEs). Two applications of system of first-order ODEs have been considered with initial condition which are arm race model and drug diffusion into human body model. The real-life application of these model has been discussed and applied in real-world. These applications have been solved by RK4 method by using different step size with the help of MATLAB. The absolute error analysed are also carried out explicitly in the framework. At each point of interval, the value of the system has been calculated and compared with its exact value at that point. Moreover, for arm race model, a comparison of the computation is compared to existing method is worked out to illustrate the general advantage of proposed method.

Keywords: Fourth-order Runge-Kutta method, system of Ordinary Differential Equations, Linear Application

1. Introduction

Ordinary differential equations (ODEs) are essential to solve problems or to gain insight into the phenomenon that arises outside mathematics. Although the basics mathematical model is discussed, the model has solved a variety and numerous complex issues. The primary method that derives is constantly improved and used by mathematicians to provide equations applicable in the modern world. Systems of ODEs play vital roles not just in mathematics but also apply in another professional field. In biology, system of ODEs is used to identify and predict the spread of disease while the diffusion of the drug into human body uses the system of ODEs to solve the pharmacology problem.

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The fourth-order Runge-Kutta (RK4) is the numerical method widely used to compute an approximation at each step of the sequences. Ten ODEs of the first-order equation with boundary conditions are solved with three methods. Comparison results show that the RK4 method is better in all cases among all in [1]. RK methods became important in studying explicit and implicit methods for solving ODEs through time discretization by [3]. RK4 technique is more accurate and the numerical solution derived by this approach converges faster. The numerical solution of the two first-order ODEs was also calculated more efficiently using the RK4 technique than by the forward Euler technique [3]. Meanwhile, [4] showed RK5 and RK8 are less efficient than RK4 since RK4 involves less computational time to compute truncation global error in the numerical solution.

1.1 Applications of system of first-order ODEs

The main focus of this paper is to study and understand the application that used a linear system of first-order ODEs and solved it numerically. Therefore, in this sub-section, we will introduce two relevant applications appropriate in the real world nowadays. In this case, we study the drug diffusion in the gastrointestinal track (GI tract) and blood. A two-compartment model for drug absorption and circulation through the GI tract and blood been formulated. Pharmacokinetics used to introduce biomathematical modelling to the students. The goals for the model are to achieve the desired effect for a medication and how long it takes to reach the desired effect in [5].

Next application that considers system of linear of first-order ODEs in the equation is economic arm race model. Arm race models can be defined as can be thought of as enduring rivalries between pairs of hostile powers which prompt competitive acquisition of military capability [6]. The phrase "arms race" was coined in the nineteenth century and was widely cited as one of the causes of World War I. He quotes Lord Grey, Britain's Foreign Secretary at the time of the conflict, as saying after the war, "Great weapons lead inexorably to war". If there are armaments on one side, there must be armaments on the other sides [7]. In [8] MacKay combines Richardson's arms race equations with Lanchester's attritional dynamics.

The objectives of this study are to study application of system of first-order ODEs by solving two mathematical models which are arm race model and drug diffusion into human body model. Next objective is solving the applications by using exact method and RK4 method. Then, the result of applications of system of first order ODEs with RK4 method is compares with its exact solution and existing method which is ZDTM method for arm race model.

2. Materials and Methods

The RK4 method is used to solve two problems regarding applications of system of first-order ODE which is drug diffusion into human body model and arm race model.

2.1 Mathematical Model

2.1.1 Drug diffusion into human body model

Theoretically regarded as a continuous function, the flow diffuses from the GI tract, denoted as $c_1(t)$ into the bloodstream, $c_2(t)$ before eliminated from the body. Assume the initial concentration of drug as c_o. The general formula of the equation is given as

$$\begin{aligned} \frac{dc_1}{dt} &= -k_1 c_1 \\ \frac{dc_2}{dt} &= k_1 c_1 - k_2 c_2 \end{aligned} \qquad Eq.1$$

with initial conditions $c_1(0) = c_0$ and $c_2(0) = 0$.

2.1.2 Arm race model

In this application of system of first order ODEs will consider two countries denoted as x and y for country 1 and country 2, respectively. Let assume that the defense spending would be decreasing at a rate proportional to the amount being spent. The two-time functions constitute the solution of the following system of ODEs as below

$$\frac{dx}{dt} = -kx + \alpha y + g \qquad Eq.2$$
$$\frac{dy}{dt} = lx - \beta y + h$$

where $x(t)$ and $y(t)$ represent the amount of weaponry at time t for country 1 and country 2 respectively, k and l represent the efficiency increasing the armaments of $x(t)$ and $y(t)$ respectively. While α and β represent the restraining factor; g and h are grievance constants.

2.2 Numerical method

2.2.1 Fourth-order Runge-Kutta Method

The RK4 method is based on [9] and described as the following

$$y_j = y_{j-1} + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = f(t_{j-1}, y_{j-1})$$
$$k_2 = f(t_{j-1} + \frac{1}{2}h, y_{j-1} + \frac{1}{2}hk_1) \qquad Eq.3$$
$$k_3 = f(t_{j-1} + \frac{1}{2}h, y_{j-1} + \frac{1}{2}hk_2)$$
$$k_4 = f(t_{j-1} + h, y_{j-1} + hk_3)$$

RK4 is a four-stage explicit Runge-Kutta method, which means that four function evaluations are needed to advance the numerical solution in a one-time step.

2.3 Exact solution

Homogeneous system of ODEs can be written as follow

$$x'_1(t) = a_{11}x_1 + a_{22}x_2 \qquad Eq.4$$
$$x'_2(t) = a_{21}x_1 + a_{22}x_2$$

where $x_n, n = 1, 2$ is a function of t.

To solve homogenous system of ODEs, it can be written in a matrix form $x' = Ax$. Where

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad Eq.5$$

and A is known as the coefficient of matrix. Then, the eigenvalues are determined using

$$|A - \lambda I| = 0 \qquad Eq.6$$

with I is an identity matrix. The roots of the characteristic's polynomial, λ_1 and λ_2 are the eigenvalues of A. To find the corresponding eigenvectors, we put each eigenvalue above into

$$(A - \lambda I)v = 0 \qquad Eq.7$$

Let $\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ and $\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$ be the eigenvectors. Therefore, the general solutions become

$$x(t) = c_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} e^{\lambda_2 t} \qquad Eq.8$$

Next, consider the system of non-homogenous as follow

$$\begin{aligned} x'_1(t) &= a_{11}x_1 + a_{12}x_2 + f_1 \\ x'_2(t) &= a_{21}x_1 + a_{22}x_2 + f_2 \end{aligned} \qquad Eq.9$$

it can be written in matrix form $x' = Ax + f(t)$, where

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad Eq.10$$

Therefore, the complementary solution is given as

$$x(t) = c_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} e^{\lambda_2 t} \qquad Eq.11$$

Then we set

$$x_p = M(t)u(t) \qquad Eq.12$$

where $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ is a fundamental matrix of a system $x' = Ax$, and let

$$M'(t) = AM(t) \qquad Eq.13$$

From Eq 12, by product rules we get

$$x'_p = M(t)u'(t) + u(t)M'(t) \qquad Eq. 14$$

substitute Eq. 14 into Eq. 10 and consider Eq. 12 and Eq. 13. Then multiplying both sides by $M^{-1}(t)$ and integrate with respect to t, we conclude that a particular solution is

$$x_p = M(t)\int M^{-1}(t) f(t)dt \qquad Eq. 15$$

Therefore, the general solution to the system is

$$x(t) = c_1\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} e^{\lambda_2 t} + x_p(t) \qquad Eq. 16$$

3. Results

The applications with initial conditions are discussed to solved the applications problem. Drug diffusion into human body model is denoted as test problem (a) which is the homogenous equation while arm race model is denoted as test problem (b) which is the non-homogenous system. The exact solution of the applications will be obtain using the method in section 2.3. The application then will be solved numerically using RK4 method. The result for RK4 method is tested using different step size, h which is *h=0.01* and *h=0.001* to determine the right step size to obtain the accuracy of RK4 method.

3.1 Test problem (a)

Consider the homogenous system

$$\begin{aligned} \frac{dc_1}{dt} &= -k_1 c_1 \\ \frac{dc_2}{dt} &= k_1 c_1 - k_2 c_2 \end{aligned} \qquad Eq. 17$$

with initial conditions $c_1(0) = 500$ and $c_2(0) = 0$.

3.1.1 Exact solution

First write the system in matrix form as follow

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} -k_1 & 0 \\ k_1 & k_e \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad Eq. 18$$

Obtain the eigenvalue by $Eq. 6$ we get $(\lambda + k_1)(\lambda + k_e) = 0$. Where, $\lambda_1 = -k_1$ and $\lambda_2 = -k_e$. For case $\lambda_1 = -k_1$ the system is

$$\begin{pmatrix} 0 & 0 \\ k_1 & -k_e + K_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad Eq. 19$$

Solving the bottom equation $k_1 v_1 + (-k_e + k_1) v_2 = 0$, and let $v_2 = 1$ we get $v_1 = \frac{k_e - k_1}{k_1}$.

For case $\lambda_2 = -k_e$, the system is

$$\begin{pmatrix} -k_1 + k_e & 0 \\ k_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad Eq.20$$

Solving the bottom equation $(k_1 + k_e) v_1 + 0 v_2 = 0$, and let $v_2 = 1$ we get $v_1 = 0$. The general solution is

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = c_a \begin{pmatrix} \frac{k_e - k_1}{k_1} \\ 1 \end{pmatrix} e^{-k_1 t} + c_b \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-k_e t} \quad Eq.21$$

We find that $c_a = 500 \frac{k_e - k_1}{k_1}$ and $c_b = 500 \frac{-k_1}{k_e - k_1}$ hence the particular solution is

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} 500 e^{(-k_1 t)} k_1 \\ \frac{k_1}{k_1 - k_e} 500 (e^{(-k_e t)} - e^{(-k_1 t)}) \end{pmatrix} \qquad Eq.22$$

3.1.2 Numerical results

The RK4 in $Eq.3$ is used to find the numerical results of test problem (a). The result are calculated using Matlab software. The performance of the fourth-order Runge Kutta method is presented in Table 1 for step size $h = 0.01$ and Table 2 for step size $h = 0.001$ and graphically depicted in Figure 1 and Figure 2.

Table 1: Numerical results of RK4 method for solving test problem (a) $h = 0.01$.

t_i /hour	Exact Solution		Fourth-order Runge Kutta Method			
			Numerical Solution		Absolute Error	
	c_1	c_2	c_1'	c_2'	$\lvert c_1 - c_1' \rvert$	$\lvert c_2 - c_2' \rvert$
0	500	0	500	0	0	0
1	188.1064636	274.8495153	188.1064636	274.8495152	1.41113E-08	3.0914E-08
2	70.76808331	323.6872861	70.76808333	323.6872861	1.06177E-08	2.86398E-08
3	26.62386778	298.3288033	26.62386779	298.3288033	5.99182E-09	2.14817E-08
4	10.01624323	253.7385503	10.01624324	253.7385503	3.00556E-09	1.55594E-08
5	3.768240186	208.8713287	3.768240188	208.8713286	1.41342E-09	1.14278E-08
6	1.417660671	169.476797	1.417660672	169.476797	6.38094E-10	8.60857E-09
7	0.533342271	136.6109062	0.533342271	136.6109062	2.80071E-10	6.62996E-09
8	0.200650257	109.7835679	0.200650257	109.7835679	1.20418E-10	5.18806E-09
9	0.075487221	88.09921423	0.075487221	88.09921422	5.09659E-11	4.10039E-09
10	0.028399268	70.65091881	0.028399268	70.65091881	2.13044E-11	3.25981E-09
11	0.010684172	56.64064108	0.010684172	56.64064108	8.8165E-12	2.60025E-09

12	0.004019524	45.40199894	0.004019524	45.40199893	3.61843E-12	2.07786E-09
13	0.001512197	36.39083117	0.001512197	36.39083117	1.47474E-12	1.66193E-09
14	0.000568908	29.16721638	0.000568908	29.16721638	5.97494E-13	1.32997E-09
15	0.000214031	23.37714252	0.000214031	23.37714252	2.40841E-13	1.06463E-09
16	8.05211E-05	18.73634104	8.05E-05	18.73634104	9.66479E-14	8.52332E-10

Table 2: Numerical results of RK4 method for solving test problem (a) with $h = 0.001$

t_i /hour	Exact Solution		Fourth-order Runge Kutta Method			
			Numerical Solution		Absolute Error	
	c_1	c_2	c_1'	c_2'	$\lvert c_1 - c_1' \rvert$	$\lvert c_2 - c_2' \rvert$
0	500	0	500	0	0	0
1	188.1064636	274.8495153	188.1064636	274.8495153	1.27898E-12	1.26989E-08
2	70.76808331	323.6872861	70.76808331	323.6872861	1.09424E-12	1.49537E-08
3	26.62386778	298.3288033	26.62386778	298.3288033	6.18172E-13	1.37827E-08
4	10.01624323	253.7385503	10.01624323	253.7385503	2.59348E-13	1.17213E-08
5	3.768240186	208.8713287	3.768240186	208.8713286	1.37668E-13	9.64877E-09
6	1.417660671	169.476797	1.417660671	169.476797	6.35048E-14	7.82856E-09
7	0.533342271	136.6109062	0.533342271	136.6109062	2.88658E-14	6.30996E-09
8	0.200650257	109.7835679	0.200650257	109.7835679	1.24623E-14	5.07106E-09
9	0.075487221	88.09921423	0.075487221	88.09921422	5.31519E-15	4.06968E-09
10	0.028399268	70.65091881	0.028399268	70.65091881	2.21004E-15	3.26372E-09
11	0.010684172	56.64064108	0.010684172	56.64064108	8.98587E-16	2.61664E-09
12	0.004019524	45.40199894	0.004019524	45.40199893	3.84241E-16	2.09746E-09
13	0.001512197	36.39083117	0.001512197	36.39083117	1.56125E-16	1.68124E-09
14	0.000568908	29.16721638	0.000568908	29.16721638	6.15827E-17	1.34747E-09
15	0.000214031	23.37714252	0.000214031	23.37714252	2.41777E-17	1.07993E-09
16	8.05211E-05	18.73634104	8.05E-05	18.73634104	9.97466E-18	8.6553E-10

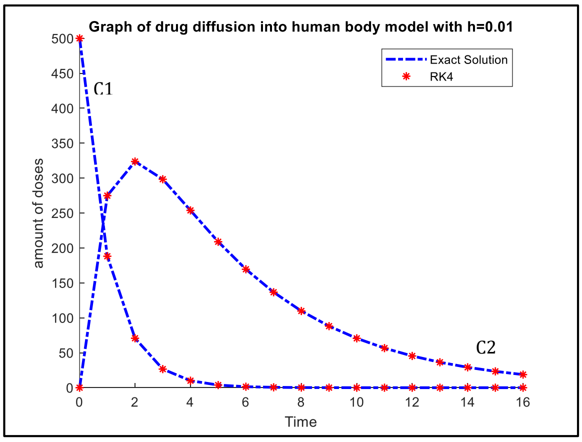

Figure 1: Graph of Drug Diffusion into human body model for exact solution and RK4 method with step size h=0.01

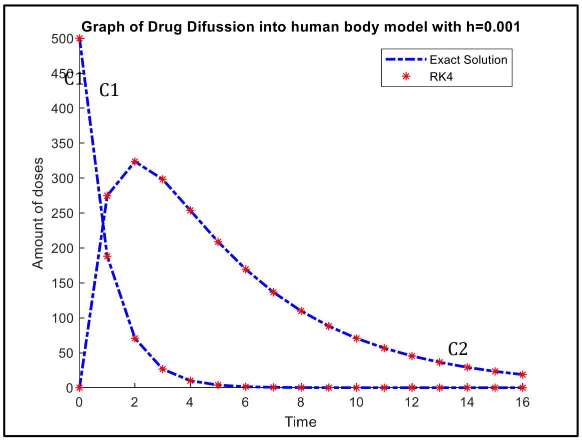

Figure 2: Graph of Drug Diffusion into human body model for exact solution and RK4 method with step size h=0.001

The drug initial doses are given as 500 and the time for the drug to completely eliminated from GI tract is calculated 16 hours. The graph in Figure 1 and Figure 2 shows that the drug absorbs into GI tract that denoted as c_1, we can see the graph decreasing over time. Meanwhile the drug eliminated from GI tract and entered bloodstream which is denoted as c_2 we can see the graph increase and at some point, it decreases and approaching zero. At last the amount of drug is totally eliminated from the body when the graph of c_1 and c_2 is equal to zero.

Meanwhile in the Table 1 and Table 2 showed the result of exact solution and RK4 method. In comparing the two solutions, we can see clearly that RK4 method is more preferable by using smaller step size.

3.2 Test problem (b)

Consider the non-homogenous system

$$\frac{dx}{dt} = 4y - 3x + 2, \quad \frac{dy}{dt} = 2x - y + 2 \qquad Eq. 23$$

with initial conditions of $x(0) = 4$ and $y(0) = 1$.

3.2.1 Exact solution

First obtain the eigenvalue from $Eq. 6$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -5$. For case $\lambda_1 = 1$,

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad Eq. 24$$

Solving the top equation $v_1 - v_2 = 0$ and let $v_1 = 1$ we get $v_1 = v_2$. For case $\lambda_2 = -5$,

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad Eq. 25$$

Solving the top equation $v_1 + 2v_2 = 0$, we get $v_1 = -2v_2$. Let $v_2 = 1$ then, $v_1 = -2$. Therefore, the complementary solution is

$$f(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-5t} \qquad Eq. 26$$

$$M(t) = \begin{pmatrix} e^{t} & -2e^{-5t} \\ e^{t} & e^{-5t} \end{pmatrix} \qquad Eq. 27$$

Find the inverse and x_p as in $Eq. 15$. we get $x_p = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Therefore, the general solution to the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4e^{t} + 2e^{(-5t)} - 2 \\ 4e^{t} - e^{(-5t)} - 2 \end{pmatrix} \qquad Eq. 28$$

3.2.2 Numerical results

The performance of the RK4 method is presented in Table 3 for step size $h = 0.01$ and Table 4 for step size $h = 0.001$ and graphically represented in Figure 3 and Figure 4. The RK4 method will be compare to exact solution and ZDTM method that obtain from [10]. In [10], the arm race model has been solved using ZDTM method to determine the efficiency of ZDTM method. Therefore, we compare the result of RK4 with the ZDTM method.

Table 3: Numerical result of RK4 method for solving test problem (b) with *h=0.01*.

t_i	Exact Solution		Fourth-order Runge Kutta Method				ZDTM Method	
			Numerical Solution		Absolute Error		Absolute Error	
	x	y	x'	y'	$\|x-x'\|$	$\|y-y'\|$	$\|x-x'\|$	$\|y-y'\|$
0	4	1	4	1	0	0	0	0
0.1	3.633744992	1.814153013	3.633745025	1.814152996	3.28985E-08	1.6504E-08	5.28E-03	3.87E-02
0.2	3.621369915	2.517731591	3.621369955	2.517731571	3.98714E-08	2.00568E-08	4.84E-02	1.47E-01
0.3	3.845695551	3.17630507	3.845695587	3.176305052	3.62145E-08	1.8308E-08	2.01E-01	2.98E-01
0.4	4.237969357	3.831963507	4.237969386	3.831963492	2.91979E-08	1.48948E-08	6.00E-01	4.33E-01
0.5	4.75905508	4.512800084	4.759055102	4.512800073	2.20138E-08	1.14157E-08	1.46E+00	4.53E-01

Table 4: Numerical result of RK4 method for solving application test problem (b) with *h=0.001*.

t_i	Exact Solution		Fourth-order Runge Kutta Method				ZDTM Method	
			Numerical Solution		Absolute Error		Absolute Error	
	x	y	x'	y'	$\|x-x'\|$	$\|y-y'\|$	$\|x-x'\|$	$\|y-y'\|$
0	4	1	4	1	0	0	0	0
0.1	3.633744992	1.814153013	3.633744992	1.814153013	3.17257E-12	1.58762E-12	5.28E-03	3.87E-02
0.2	3.621369915	2.517731591	3.621369915	2.517731591	3.83604E-12	1.93756E-12	4.84E-02	1.47E-01
0.3	3.845695551	3.17630507	3.845695551	3.17630507	3.48743E-12	1.76303E-12	2.01E-01	2.98E-01
0.4	4.237969357	3.831963507	4.237969357	3.831963507	2.80309E-12	1.43885E-12	6.00E-01	4.33E-01
0.5	4.75905508	4.512800084	4.75905508	4.512800084	2.12008E-12	1.09335E-12	1.46E+00	4.53E-01

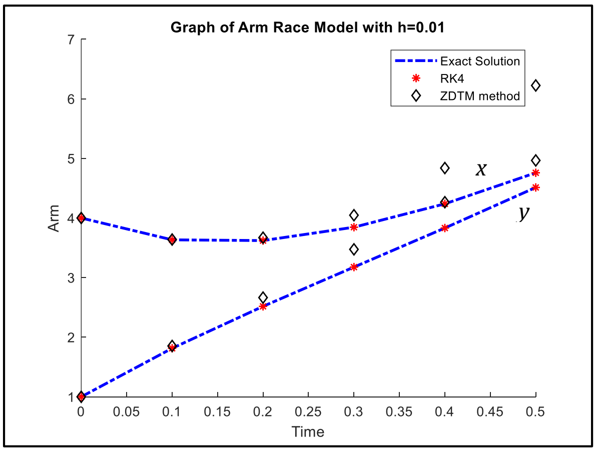

Figure 3: Graph of Arm race model for exact solution, RK4 method and ZDTM method for x and y.

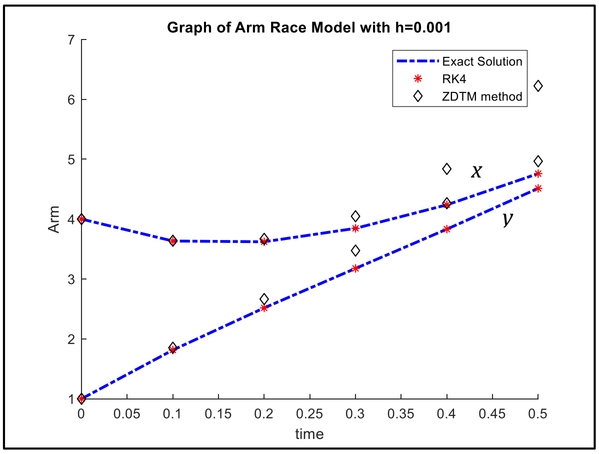

Figure 4: Graph of Arm Race model for exact solution, RK4 method and ZDTM method for x and y.

The arm race model solution in Figure 3 and Figure 4 shows that the graph the country 1 denoted as x increasing their armaments. In a beginning, country 2 denoted as y will decrease their defences spending. When x increase the spending for defences, y will see the needs to increase the spending. Meanwhile for the numerical results in Table 3 and Table 4, comparing the two methods, we can see clearly that RK4 method is more preferable than ZDTM method and the result of RK4 method with smaller step size gives better approximations to the exact solutions. If only ZDTM increasing the order of approximate more accurate solution but its time consuming and tedious to apply.

4. Conclusion

The RK4 method has very effective approach for many applications in various field of science and engineering. By comparing exact solution and RK4 method for both applications of system of ODEs, RK4 method is approximately accurate since its approaching to exact solution. The result with smaller step sizes gives better result and less absolute error compare to larger step size. Meanwhile, in application of system of ODEs of arm race model the result of RK4 is more accurate approaching to exact value than ZDTM method. To obtain the approximate result, ZDTM method need to increasing the order which is tedious and time consuming for computational work. In this research, RK4 solved the arm race model and drug diffusion into human body model which reduces the computational work than other traditional method.

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