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Solving Higher Order Ordinary Differential Equation (ODE) by using Differential Transformation Method (DTM)

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Abstract: The ordinary differential equation (ODE) is important in the mathematical study. Commonly, higher-order ODEs will be used to form the formulation of a real-world problem. But the higher-order ODEs are complex to solve. Therefore, the differential transformation method (DTM) has been introduced. The DTM is semi numerical method that resolves the equation in the form of a polynomial. In this study, two different higher orders of ODEs have been solved by using DTM. The numerical solution derived by DTM has been compared with the numerical solution from the Adomian decomposition method (ADM) and the exact solution to see the accuracy of the method. By the end of the research, the accuracy of the DTM was proved to solve the higher-order ODEs with the minimum absolute error compared to the other method.

Keywords: Differential Transformation Method (DTM), Higher Order Ordinary Differential Equation (ODE)

1. Introduction

In 1986, the differential transformation method (DTM) concept is originally presented by Zhou for solving initial value problems in electric circuit analysis [1]. Later, this method has been used to solving the various types of equations. The DTM is a half numerical also known as an analytical approach to solving the problem of ordinary differential equations (ODEs) [2], partial differential equations (PDEs) [3] and fractional differential equation (FDE) [4]. The DTM resolves the equation in the form of a polynomial same as the Taylor series method. Therefore, most of the research confirms that this method is reliable, efficient, and has wider applicability to solve any type of equations [5].

In this study, the higher order ODEs has been chosen to solve by using DTM. ODE is a differential equation that implies some ordinary derivatives. ODE has been applied in many areas including physics, engineering and population dynamics. In real life, ODEs are used to calculate the movement or flow of

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electricity, to explain thermodynamics concepts and the growth of diseases in medical terms. The ODEs contain many levels of order such as first, second and higher-order derivatives. This order is referring to the highest order derivative in the equation. Commonly, higher order ODE will be used to form the formulation of a real-world problem.

The purpose of this study is to explore the DTM for solving higher order ordinary differential equation (ODE), to obtain the analytical solution of given higher order ODE by DTM and to compare the accuracy of DTM with other method.

2. Methodology

2.1 Higher Order Ordinary Differential Equation

Consider the equation of higher order ODE as follows

$$y^{n}(x)+a_{1}y^{n-1}(x)+...+a_{n-1}y'(x)+a_{n}y(x)=0 \qquad \text{Eq. 1}$$

where $a_n, a_{n-1}, ..., a_1$ are real constants with the initial conditions;

$$y(o)=y_0, y'(0)=y_1, ..., y^{(n-1)}(0)=y_{(n-1)} \qquad \text{Eq. 2}$$

where $y_0, y'_0, ..., y_0^{(n-1)}$ are real constants.

2.2 Differential Transformation Method

Define the differential transform method of a function $y(x)$ as

$$Y(k)=\frac{1}{k}\left[\frac{d^{k}y(x)}{dx^{k}}\right] \qquad \text{Eq. 3}$$

where $Y(k)$ is a transformed function. The inverse of $Y(k)$ is express as

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k} \approx Y_{N}(x)=\sum_{k=0}^{N}Y(k)x^{k} \qquad \text{Eq. 4}$$

Table 1 listed the fundamental mathematical operations of the differential transformation.

Table 1: The fundamental mathematical operations. [6]

Original function	Transformed function
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=\alpha u(x)$	$Y(k)=\alpha U(k)$
$y(x)=y'(x)$	$Y(k)=(k+1)Y(k+1)$
$y(x)=y^{n}(x)$	$Y(k)=\frac{(k+n)!}{k!}Y(k+n)$
$y(x)=x^{n}(x)$	$Y(k)=\delta(k-n)=\begin{cases}1 if k=n\\0 if k=0\end{cases}$

$y(x)=e^{\lambda k}$	$Y(k)=\dfrac{\lambda^k}{k!}$

By applying the fundamental mathematics equations in Table 1 on Eq. 1 and Eq. 2, we get

$$\frac{(k+n)!}{k!}Y(k+n)+a_1\left[\frac{(k+n-1)!}{k!}Y(k+n-1)\right]+...+a_{n-1}\left[(k+1)Y(k+1)\right]+a_n\left[Y(k)\right]=0 \quad \text{Eq. 6}$$

where $k=1,2,...,m$ and $n=1,2,...,l$ and

$$Y(0)=y_0, Y(1)=y_1,...,$$

$$Y(m+n)=-\frac{1}{(m+n)!}\left[a_1\left[(m+n-1)!Y(m+n-1)\right]+...+a_{n-1}\left[(m+1)Y(m+1)\right]+a_nY(m)\right] \quad \text{Eq. 7}$$

Lastly, the series solution is obtained

$$Y(x)=Y(0)x^0+Y(1)x^1+...+Y(n)x^n \quad \text{Eq. 8}$$

3. Results and Discussion

3.1. Example 1

Consider the first equation is third order ODEs [7]

$$y'''=-y \quad \text{Eq. 9}$$

with the initial conditions

$$y(0)=1, y'(0)=-1, y''(0)=1 \quad \text{Eq. 10}$$

where the exact solution is given by

$$y=e^{-x}. \quad \text{Eq. 11}$$

By applying DTM to Eq. 9, we obtain

$$\frac{(k+3)!}{k!}Y(k+3)=-Y(k) \quad \text{Eq. 12}$$

and

$$Y(0)=1, Y(1)=-1, Y(2)=\frac{1}{2}. \quad \text{Eq. 13}$$

Applying the transformed initial condition Eq. 13 in Eq. 12, we obtain the following

$$Y(3)=-\frac{1}{6}, Y(4)=\frac{1}{24}, Y(5)=-\frac{1}{120}, Y(6)=\frac{1}{720}, Y(7)=-\frac{1}{5040}.$$

We finally obtain the series solution up to seventh term as

$$Y(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7. \quad \text{Eq. 14}$$

Table 2: Numerical solution of DTM, ADM and Exact solution.

x	Exact solution	DTM solution	ADM solution
0	1.0000000000	1.0000000000	1.0000000000
0.1	0.9048374180	0.9048374181	0.9048374181
0.2	0.8187307531	0.8187307531	0.8187307556
0.3	0.7408182207	0.7408182191	0.7408182625
0.4	0.6703200460	0.6703200305	0.6703203556
0.5	0.6065306597	0.6065305680	0.6065321181
0.6	0.5488116361	0.5488112457	0.5488168000
0.7	0.4965853038	0.4965839780	0.4966003181
0.8	0.4493289641	0.4493251454	0.4493667556
0.9	0.4065696597	0.4065599623	0.4066548625
1.0	0.3678794412	0.3678571429	0.3680555556

Table 3: The absolute error of DTM, ADM and the minimum absolute error to the exact solution

x	DTM	ADM	Minimum
0	0	0	0
0.1	1E-10	1E-10	1E-10
0.2	0	2.5E-09	0
0.3	1.6E-09	4.18E-08	1.6E-09
0.4	1.55E-08	3.096E-07	1.55E-08
0.5	9.17E-08	1.4584E-06	9.17E-08
0.6	3.904E-07	5.1639E-06	3.904E-07
0.7	1.3258E-06	1.50143E-05	1.3258E-06
0.8	3.8187E-06	3.77915E-05	3.8187E-06
0.9	9.6974E-06	8.52028E-05	9.6974E-06
1.0	2.22983E-05	1.76114E-04	2.22983E-05

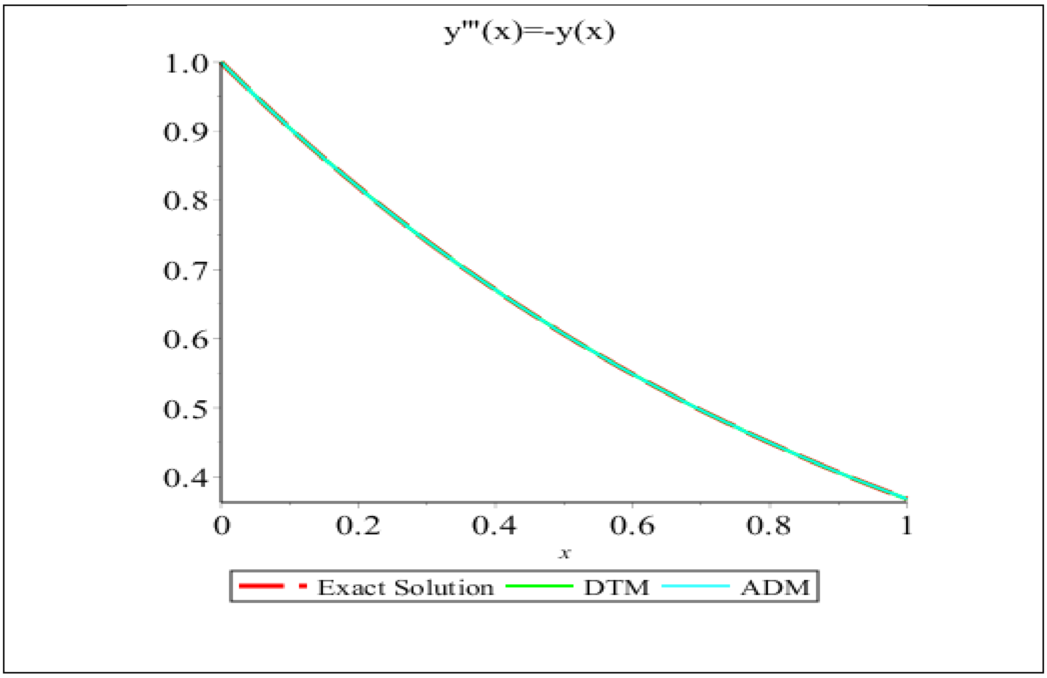

Figure 1: The graph of numerical solution for exact solution, DTM and ADM

Figure 1 shows the graph or numerical solution for the exact solution, DTM, and ADM obtains by using MAPLE software. The trend of the graph for DTM and ADM is slightly the same as the exact solution. In Table 3, the absolute error for DTM and ADM shows that the numerical solution for both methods is close to the exact solution. But, by comparing the absolute solution for all the method, the smallest absolute error goes to DTM for $0 \le x \le 10$. This show that DTM are more accurate than ADM to the exact solution using absolute error.

3.2 Example 2

Consider the fifth order ODEs [8]

$$y^{(5)} + 5y^{(4)} + 10y''' + 10y'' + 5y' + y = 32e^{x} + e^{4x} - y^{4} \qquad \text{Eq. 15}$$

with the initial conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 1 \qquad \text{Eq. 16}$$

where the exact solution is given by

$$y = e^{x}. \qquad \text{Eq. 17}$$

By applying DTM to Eq. 15, we obtain

$$\begin{aligned} &\frac{(k+5)!}{k!}Y(k+5) + 5\frac{(k+4)!}{k!}Y(k+4) + 10\frac{(k+3)!}{k!}Y(k+3) + 10\frac{(k+2)!}{k!}Y(k+2) \\ &+5(k+1)Y(k+1) + Y(k) = \frac{32\times 1^{k}}{k!} + \frac{4^{k}}{k!}\delta(k) - \sum_{r=0}^{k} Y(r)Y(k-r) \end{aligned} \qquad \text{Eq. 18}$$

and

$$Y(0) = 1, Y(1) = 1, Y(2) = \frac{1}{2}, Y(3) = \frac{1}{6}, Y(4) = \frac{1}{24}. \qquad \text{Eq. 19}$$

Applying the transformed initial condition Eq. 19 in Eq. 18, we obtain the following

$$Y(5)=\frac{1}{120},Y(6)=-\frac{1}{720},Y(7)=\frac{1}{630}.$$

We finally obtain the series solution up to seventh term as

$$Y(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5-\frac{1}{720}x^6+\frac{1}{630}x^7. \quad \text{Eq. 20}$$

Table 4: Numerical solution of DTM, ADM and Exact solution.

x	Exact solution	DTM solution	ADM solution
0	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105170916	1.105178459
0.2	1.221402758	1.221402598	1.221511914
0.3	1.349858808	1.349857085	1.350358644
0.4	1.491824698	1.491815579	1.493252593
0.5	1.648721271	1.648688617	1.651867869
0.6	1.822118800	1.822027634	1.827993554
0.7	2.013752707	2.013538737	2.023511890
0.8	2.225540928	2.225099459	2.240372988
0.9	2.459603111	2.458779339	2.480559126
1.0	2.718281828	2.716865080	2.746031747

Table 5: The absolute error of DTM, ADM and the minimum absolute error to the exact solution

x	DTM	ADM	Minimum
0	0	0	0
0.1	2E-09	7.541E-06	2E-09
0.2	1.6E-07	1.09156E-04	1.6E-07
0.3	1.723E-06	4.99836E-04	1.723E-06
0.4	9.119E-06	1.427895E-03	9.119E-06
0.5	3.2654E-05	3.146598E-03	3.2654E-05
0.6	9.1174E-05	5.874746E-03	9.1174E-05
0.7	2.1397E-04	9.759183E-03	2.1397E-04
0.8	4.41469E-04	1.483206E-02	4.41469E-04
0.9	8.23772E-04	2.0956015E-02	8.23772E-04
1.0	1.416748E-03	2.7749919E-02	1.416748E-03

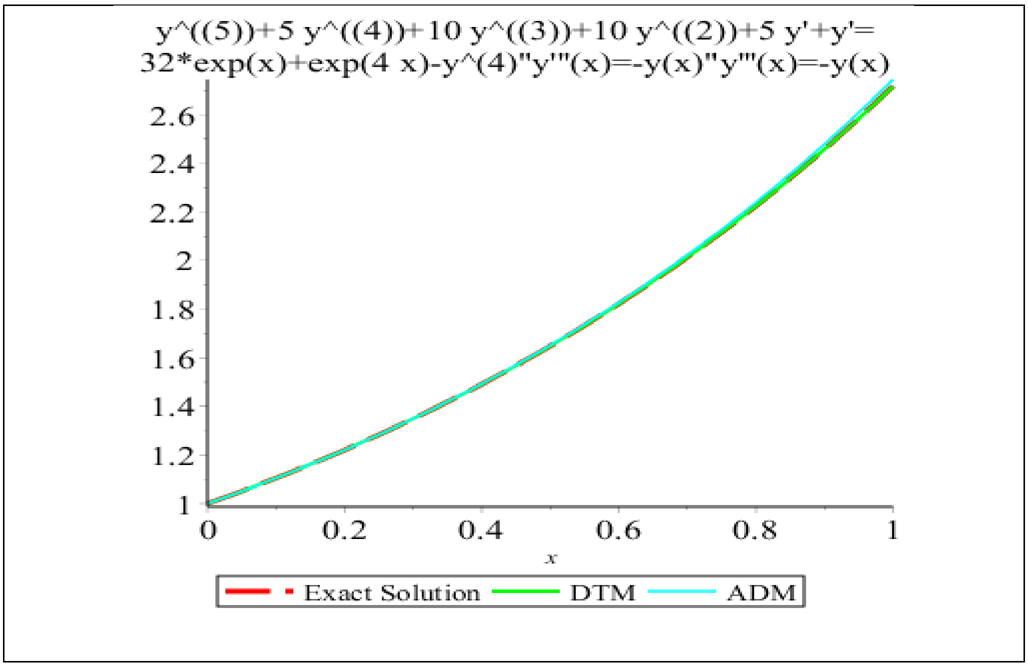

Figure 2: The graph of numerical solution for exact solution, DTM and ADM

Figure 2 shows the graph of numerical solution for exact solution, DTM, and ADM that obtained by using Maple software. The trend of the graph for DTM and exact solution is similar but graph for ADM is slightly diverge from the exact solution. By comparing the absolute error for both methods in Table 5, the DTM has the smallest absolute error compared to ADM for $0 \le x \le 10$. This shows that the numerical solution for DTM is closer to the exact solution compared to the ADM.

4. Conclusion

In this study the background of the higher order ODEs and DTM have been explored. By using DTM, two different higher order of ODEs have successfully solved with the help of Maple software and Excel. The numerical result for the both examples had been compared to the ADM to know the accuracy of this method to the exact solution. The results show that DTM given more accurate solution with the minimum absolute error. To improve the accuracy of the result, the equations can be solve by using modified differential transform method with Adomain polynomials or Laplace transform or Padé approximation [5].

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