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Boundary Layer Flow Over a Vertical Plate

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Abstract: The focus of this research is to study the nonlinear boundary layer problem by considering the natural convection of heat transfer about a vertical plate. The nonlinear coupled partial differential equations of momentum and energy equation are transformed into nonlinear ordinary differential equation using similarity transformation. This study highlights a comparison between two different numerical methods in solving two-point boundary layer problem to analyze the reliability and efficiency of each method. The method used in this research is bvp4c method which is one of the built-in programs in MATLAB for solving boundary layer problem and the comparison are made with shooting method from the previous research. The velocity distributions and temperature distributions are illustrated. The result obtained shows a good agreement between two methods in solving the boundary layer problem.

Keywords: Boundary Layer Problem, Similarity Transformation, bvp4c Method

1. Introduction

The boundary-layer theory began in August 1904 which presented at the Third International Congress of Mathematicians by Ludwig Prandtl. His paper entitled 'Motion of a fluid with very small viscosity' are then published in 1905 in the Proceedings of the Congress [1]. Ludwig Prandtl has made significant contributions in the fluid mechanics and dynamics field through his paper. Among the topic discussed in his paper are the fluid motions against solid surface with low viscosity.

Through his paper, Ludwig Prandtl proved that viscosity cannot be neglected no matter how small its viscosity. Newton's shear stress law states that the shear stress is linearly proportional to the velocity gradient, and this causes the drag force exerted on the body by skin friction is not insignificant. Hence, Ludwig Prandtl denied the beliefs of some earlier nineteenth century researchers regarding the negligence drag force due to the skin friction [2]. A dimensionless measure introduced by George Stokes known as Reynolds number which define as the ratio of inertial forces to viscous forces within fluid provides insight on pattern of the fluid flow.

In general, the boundary layer defined as the flow region near the solid surface in which the flow is slowed by the viscosity of the fluid. When the fluid interacts with the surface of flat plate, a region in the fluid formed where the fluid flow experience viscosity effect. The viscosity is a measure of a fluid

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resistance to flow which describes the internal friction force arises between a fluid and a surface that are in relative motion. When in motion, low viscosity fluid indicates that the fluid flow easily due to little friction compared high viscosity fluid with higher friction. The fluid with low viscosity values has high Reynolds number. Low Reynolds number indicates that laminar flow occurs where the fluid flow smoothly while high Reynolds number results in turbulent flow occurs due to instabilities of the fluid flow [3].

There are many studies have emerged after Ludwig Prandtl which deal with boundary layer problem by making his boundary layer theory as a reference after the first paper published by Ludwig Prandtl. One of the popular studies are proposed by Heinrich Blasius in 1908. Heinrich Blasius derived the fluid profile for two-dimensional boundary layer flow over a flat plate with no pressure gradient which written in his paper entitled 'Grenzschichten in Flüssigkeiten mit kleiner Reibung' ('The Boundary Layers in Fluids with Little Friction'). The study was extended by E. Pohlhausen where he proposed a solution of boundary layer problem for thermal boundary layer over the flat plate [4].

One of the biggest branches in boundary layer of fluid dynamics field is heat transfer. The use of this study is significant and very important especially in this modern world. The first exploratory of heat transfer on a vertical plate were started in the late nineteenth century. L. Lorenz is the first researcher achieved the analytical solution of natural convection problem regarding the heat transfer between a vertical wall and air surrounding based on several assumption but in 1909, W. Nusselt proved the result does not accurately predict and not applicable when the temperature approached zero [5].

Further research regarding analytical solution of natural convection proposed by many researchers. In 1930, E. Schmidt and W. Beckmann resolved the analytical solution of L. Lorenz in 1881 with the assist of E. Pohlhausen by introducing a stream function and similarity variables which allowed the equations to be reduced and solved numerically [6]. The are many research proposed by different researchers after E. Schmidth and W. Beckmann with different defined stream function and similarity variables to simplify the governing equation of natural convection.

There are many types of numerical methods that can be used for solving boundary layer problem of PDE other than shooting method. The most common methods used by many researchers are finite difference method, finite element method and more. However, in this research, the method that will be used is bvp4c method. The use of the bvp4c method has become one of the alternatives for many researchers to solve the boundary layer problem. Bvp4c is one of the boundary value problems solving package which available in MATLAB software. It is highlighted that this method not only easy to use but also reduce the run time makes the method more advantageous than other method [7].

2. Research Methodology

The boundary layer flow over a vertical plate can be demonstrated by Figure 1. When the fluid interacts with the surface of flat plate, a region in the fluid formed where the fluid flow experience viscosity effect. In velocity boundary layer, the velocity, u rises from zero at the surface to an asymptotic value u_∞. Velocity gradient or velocity boundary layer thickness is represented by δ_h. In thermal boundary layer, the temperature rises at $y = 0$, where the value of temperature represented by T_w rises to an asymptotic value of T_∞ while the temperature gradient or thermal boundary layer thickness is represented by δ_T [8].

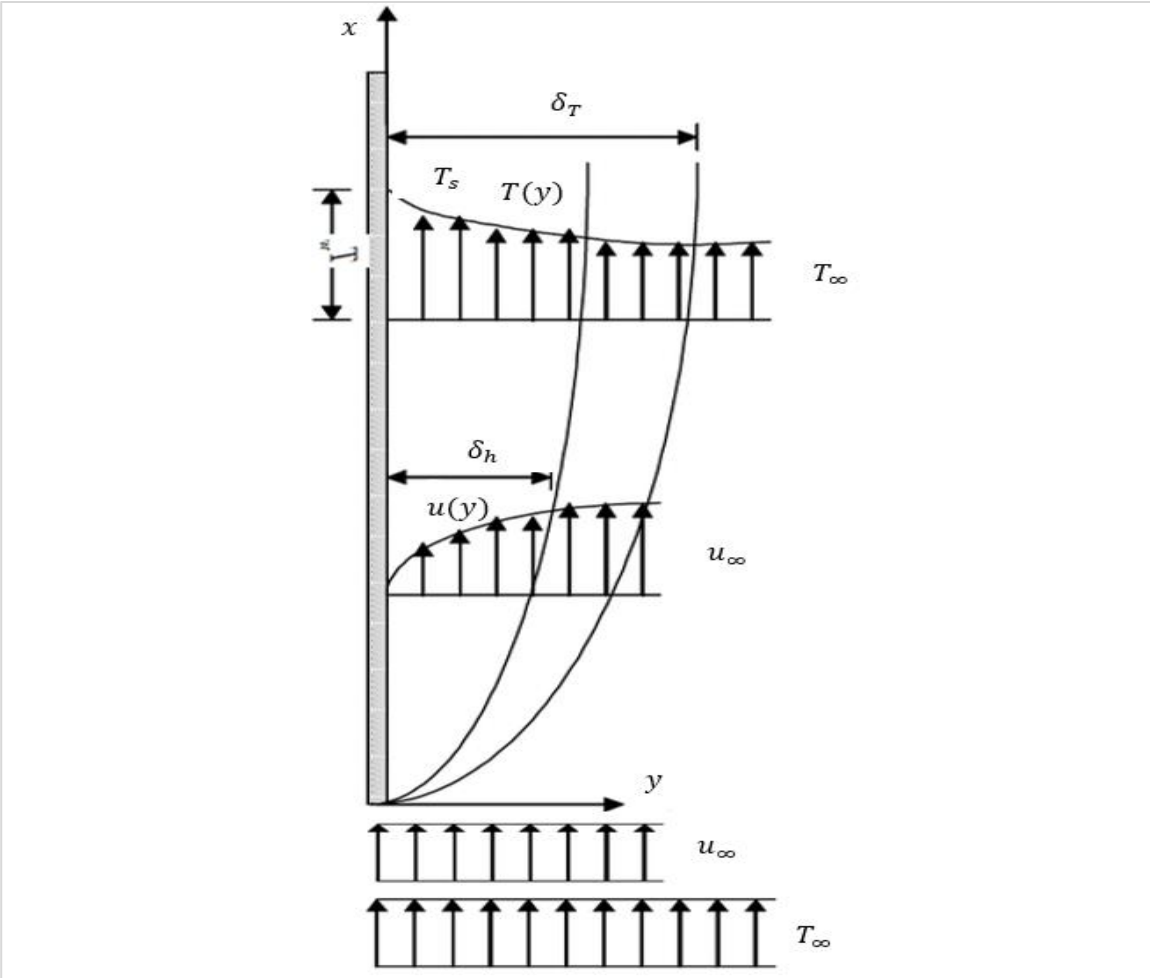

Figure 1: The boundary layer over a vertical plate [9]

The focus of this study is to solve the PDE in boundary layer problem using bvp4c method. Most problems related to PDE are too complicated to be solved analytically and need to be solved by a numerical method. In the previous study, improved shooting method is used by researchers to generate the numerical solution of boundary layer over a vertical plate [10]. In this study, another approach is proposed which is bvp4c method to compare and verify the reliability and efficiency of each numerical method in solving the two-point nonlinear boundary value problem of coupled equation of momentum and energy equation.

2.2 Governing Equation

Consider a two-dimensional steady incompressible fluid flow where the fluid moving over the vertical plate with the uniform velocity U_∞ at temperature T_∞ as shown in Figure 2. The governing equation of momentum and energy equations according to [10] are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\sqrt{Gr_x}}\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \qquad Eq. 1$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr\sqrt{Gr_x}}\frac{\partial^2 T}{\partial y^2} \qquad Eq. 2$$

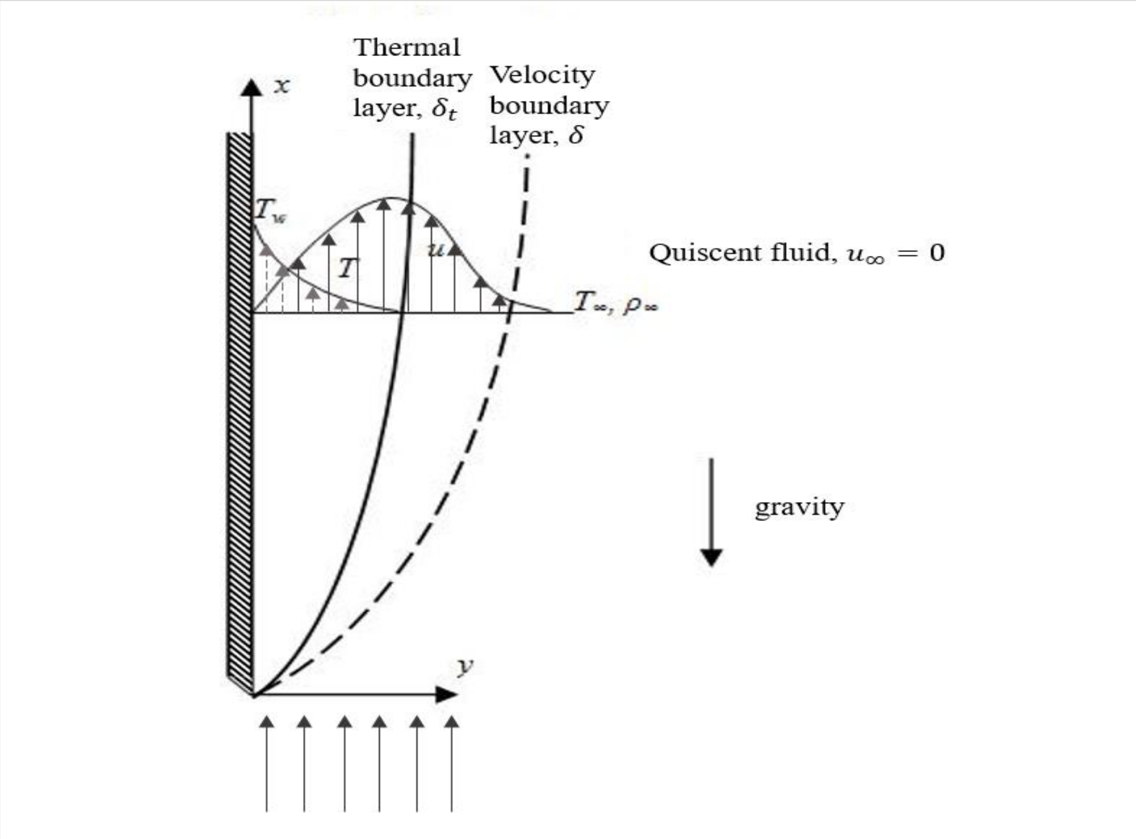

Figure 2: Natural convection over a vertical plate [11]

where Pr is the Prandtl number and Gr is the Grashof number. The boundary conditions for this problem as stated by [10] are:

$$u = v = 0, \qquad T = T_w, \qquad \text{when } y = 0 \qquad Eq. 3$$

$$u \to 0, \qquad T \to T_\infty, \qquad \text{when } y \to 0 \qquad Eq. 4$$

$$u = 0, \qquad T = T_W, \qquad \text{when } y = 0 \qquad Eq. 5$$

where T_w represents temperature on the surface of vertical plate while T_∞ represents temperature on the surrounding. Next, proceed with the analysis of natural convection by introducing a stream function, ψ, dimensionless temperature, θ, and a similarity variable, η as:

$$u = \frac{\partial \psi}{\partial y} \qquad Eq. 6$$

$$v = -\frac{\partial \psi}{\partial x} \qquad Eq. 7$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \qquad Eq. 8$$

$$\eta = \frac{y}{x}\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \qquad Eq. 9$$

where the stream function, ψ, written in terms of a function of η as:

$$\psi = v\left[4\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}\right]f(\eta) \qquad Eq. 10$$

The Grashof number, Gr, can be defined as:

$$Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{v^2} \qquad Eq. 11$$

By using Eq. 6 to Eq. 10, the coupled PDE equation of Eq. 1 and Eq. 2 are transformed into a set of coupled nonlinear ordinary differential equations as follow:

$$f''' + 3ff'' = 2f'^2 + \theta \qquad Eq. 12$$

$$\theta'' + 3Prf\theta' = 0 \qquad Eq. 13$$

The boundary conditions for a pair of nonlinear ordinary equations for the velocity and temperature are:

$$f = 0, \quad f' = 0, \quad \theta = 0, \quad \text{when } \eta = 0 \qquad Eq. 14$$

$$f' = 0, \quad \theta = 0, \quad \text{when } \eta \to 0 \qquad Eq. 15$$

2.3 Bvp4c Method

Bvp4c is one of the built-in programmes in MATLAB for solving boundary layer problem. According to [7], bvp4c is a three-stage Labatto IIIa formula implemented by finite difference method and the mesh selection based on the residual of the continuous solution. Since making a good guess for the solution are hard, bvp4c able to control error by taking unusual approach to deal with various guesses [7].

In order to execute bvp4c method to solve the third order of Eq.12 and Eq.13 obtained after the transformation of partial differential equations, the further reduction of the equations to first order system of ordinary differential equations are needed by defining new variable as follows:

$$f(\eta) = y(1) \qquad Eq. 16$$

$$f'(\eta) = y(2) \qquad Eq. 17$$

$$f''(\eta) = y(3) \qquad Eq. 18$$

$$\theta(\eta) = y(4) \qquad Eq. 19$$

$$\theta'(\eta) = y(5) \qquad Eq. 20$$

$$f'''(\eta) = -3y(1)y(3) + 2y(2)^2 - y(4) \qquad Eq. 21$$

$$\theta''(\eta) = -3Pry(1)y(5) \qquad Eq. 22$$

The new boundary conditions of the first order ordinary differential equation when $\eta = 0$ are:

$$y(1) = 0 \qquad Eq. 23$$

$$y(2) = 0 \qquad Eq. 24$$

$$y(4) = 1 \qquad Eq. 25$$

The new boundary conditions of the first order ordinary differential equation when $\eta \to \infty$ are:

$$y(2) = 0 \qquad Eq. 26$$

$$y(4) = 0 \qquad Eq. 27$$

3. Results and Discussion

This research focus on boundary layer flow over a vertical plate where the solution is generated numerically by using built-in programmes in MATLAB for solving boundary layer problem known as bvp4c. The coupled partial differential equation of Eq.1 and Eq.2 are transformed to system of ordinary differential equations by using defined similarity variable in the Eq.8 and stream function in the Eq.9. The higher order system of ordinary differential equations obtained are then reduce to first order system of ordinary differential equations. The new defined equation is then solved by bvp4c method.

The numerical calculation has been carried out for various number of Prandtl number to compare not only the reliability and efficiency of the proposed numerical method, but also to investigate the effect of Prandtl number on the velocity and temperature distribution in boundary layer flow. The Prandtl number used in this research are $0.1, 0.72, 1.0, 2.0, 5.0,$ and 7.0. Table 1 shows values of skin friction, f'', and local Nusselt number, -θ' based on different Prandtl number at of $\eta = 0$. The negative sign in the temperature gradient, θ' appears due to heat flows in the direction of decreasing temperature.

Table 1: Numerical solution based on Prandtl number

Pr	$f''(0)$	-$\theta'(0)$
0.1	0.8501	0.2357
0.72	0.6760	0.5046
1.0	0.6422	0.5671
2.0	0.5713	0.7165
5.0	0.4818	0.9539
7.0	0.4507	1.0542

From the result obtained, it shows that at $\eta = 0$. the value of skin friction, f'' decreases while the value of local Nusselt number, $-\theta'$ increases when the number of Prandtl increase. This indicates that, as the number Prandtl increases, the thermal boundary layer will be thinner resulting in lower average temperature within the boundary layer. In other word, as the number of Prandtl increases, the skin friction decrease, the velocity of thermal distribution become faster while the temperature distribution decreases.

Figure 3 illustrates the velocity distribution while Figure 4 shows the temperature distribution for various Prandtl number. The velocity is zero while the temperature is maximum at the plate surface. Away from the plate, it is noticeable that the velocity increases gradually while the temperature decreases exponentially to zero value which satisfies the boundary conditions. Figure 3 and Figure 4 shows that a Prandtl number affects velocity boundary layers and temperature distribution profile.

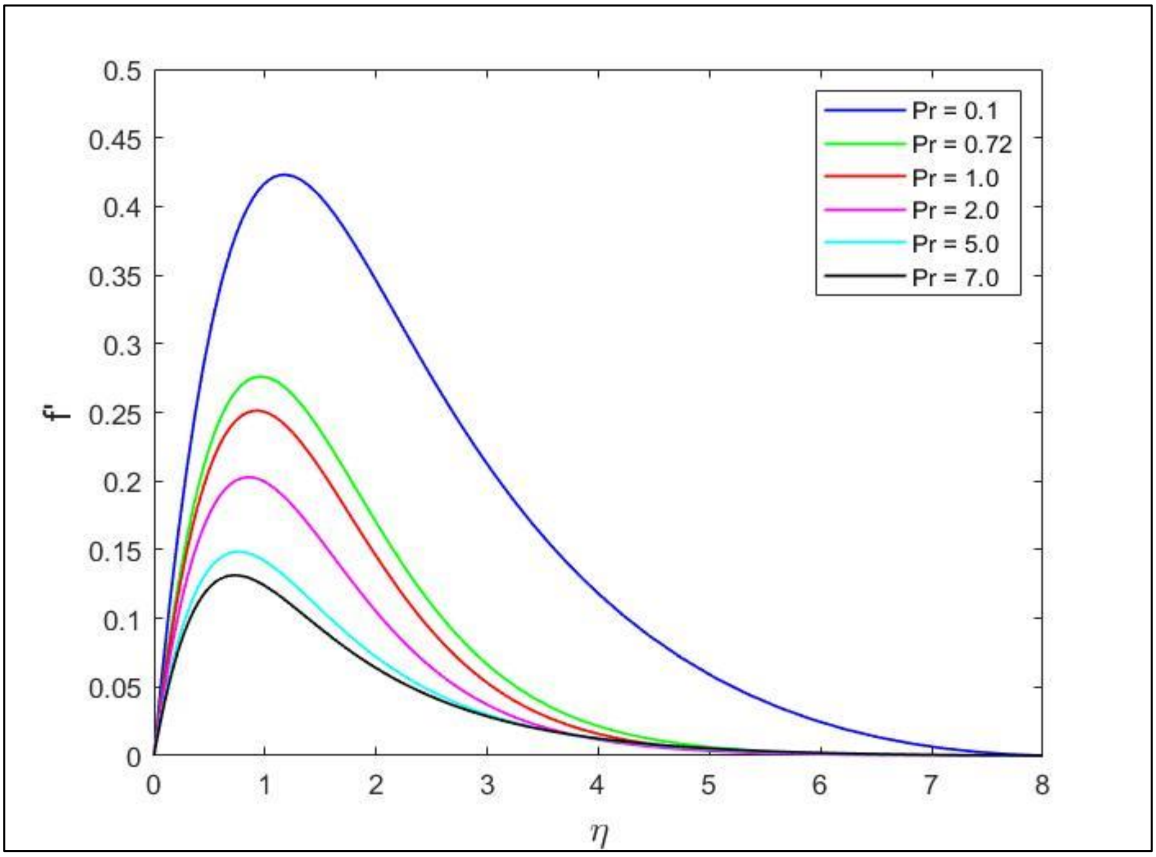

Figure 3: Velocity distribution for different values of Pr

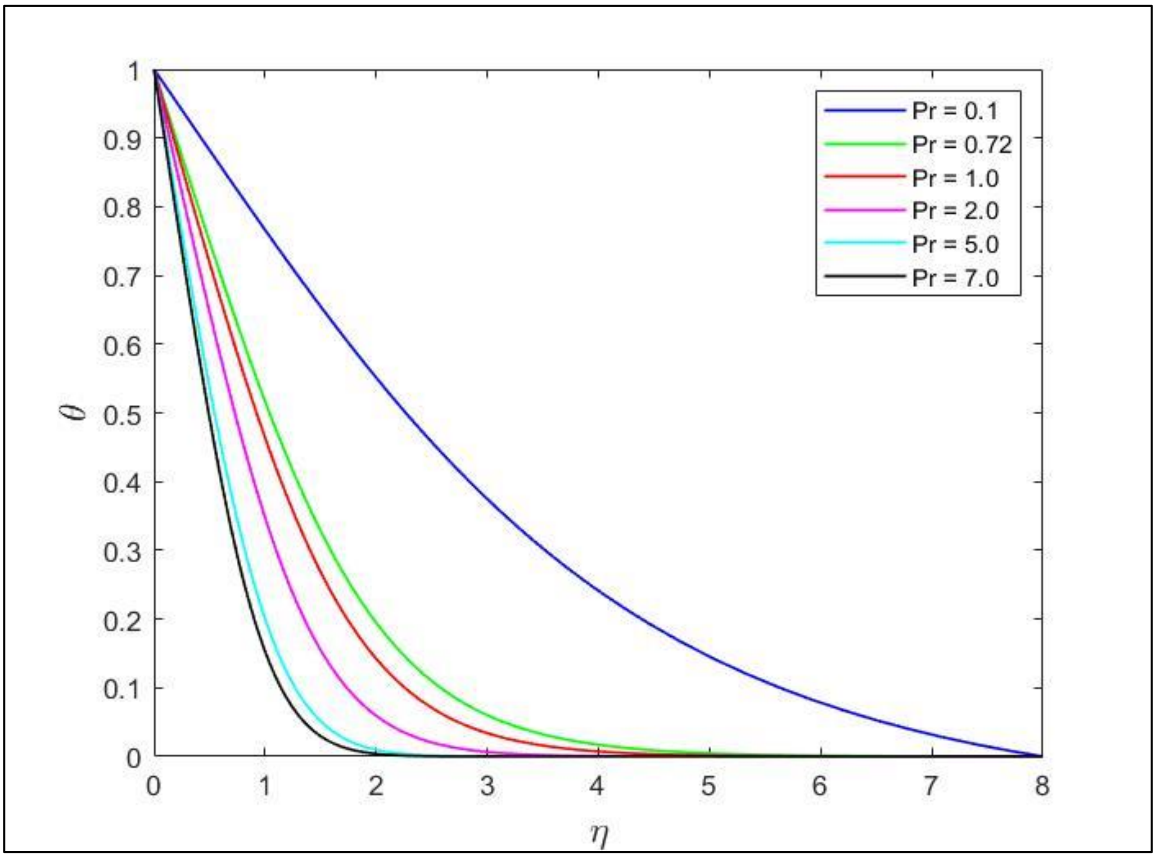

Figure 4: Temperature distribution for different values of Pr

Heat transfer rate is one of the important qualities other than velocity and temperature distribution when discussing about boundary layer flow. Table 2 shows the comparison of heat transfer parameter of present result which obtain by using bvp4c method compared to [10] which obtain by improved shooting method. From the result obtained, it shows there is slight difference from present result for Prandtl number of 0.1,2.0,5.0, and 7.0. However, the result is still in a good agreement with previous research [10].

Table 2: Comparison of heat transfer parameter

Pr	$Nu/Gr_x^{\frac{1}{4}}$		
	Present Result	Liancun et al. (2007) [10]	Gebhart, (1961)
0.1	0.167	0.165	0.164
0.72	0.357	0.357	0.357
1.0	0.401	0.401	0.401
2.0	0.507	0.506	0.507
5.0	0.675	0.673	0.675
7.0	0.745	0.737	0.754

4. Conclusion

From this research, it concluded that:

i) The bvp4c method is one of the accurate and efficient method in solving boundary layer problem since there is a good agreement when comparing the bvp4c method and improved shooting method conducted by Liancun et al. (2007) [10].
ii) The Prandtl number affects the velocity distribution and the temperature distribution of boundary layer flow.
iii) Increasing the Prandtl number leads to an increase of the velocity distribution.
iv) Increasing the Prandtl number reduces the temperature distribution.

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