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Modulation of Nonlinear Waves in Viscous Fluid Contained in Stenosed Elastic Tube

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Abstract: The purpose of this study is to investigate the non-linear wave modulation in a stenosed elastic tube filled with viscous fluid. The artery is considered as an incompressible, pre-stressed, thin-walled and long elastic tube with a symmetrical stenosis while the blood is assumed as an incompressible viscous fluid. The stretched coordinates and asymptotic series were introduced to the non-dimensional equations of tube and fluid. By implementing the method of Reductive Perturbation (RPM), a sets of non-linear differential equations of various orders is obtained. Solving these differential equations resulted in the dissipative non-linear Schrodinger (NLS) equation with variable coefficients. Analytical solutions for the dissipative NLS equation with variable coefficients is carried out. Based on the graphical output, it is noticed that, when blood flowing in a stenosed elastic tube, the radial displacement decreases gradually due to the resistance of fluid flow. It is observed that increase the blood viscosity caused an increase in the pressure to walls of arteries consist stenosis. Besides, it is found that the wave amplitude decreases obviously when the viscous effect of fluid increases. Other than that, the wave speed also increase rapidly since the cross-sectional area of artery reduced due to the existence of the stenosis.

Keywords: Wave Modulation, Dissipative Non-linear Schrodinger Equation With Variable Coefficients, Viscous Fluid, Thin Elastic Tube, Stenosed, Reductive Pertubation Method

1. Introduction

In general, blood plays an important role in delivering oxygen and nutrients to cells and transporting waste from cells in a human body so that it could maintain homeostasis and regulate the body's systems. A human body contains about 5 liters of blood and it also takes around 20 to 30 seconds to make a complete cycle through the circulation and return to the heart once it is pumped out [1]. According to Jarvis and Saman [2], it is part of the cardiovascular system, along with the heart, which acts as a pump. Besides, blood is also recognised as an incompressible non-Newtonian fluid in human body. Blood flow from arteries to the middle part of the artery reduced the hematocrit level. As the

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hematocrit ratio is low, the shear rate in the artery is high. Rudinger [3] discovered that when blood included a low level of hematocrit and a high shear rate, it behaved similarly to a viscous fluid. According to Anthony and Raja [4], it is stated that the arteries play an important role in the circulatory system. It consists of three layers which are the intima, the media, and the adventitia. A stenosis defines as the deposits of fatty substances, cholesterol, cellular waste products, calcium and other substances build up in the inner lining of an artery which can causes blockage in arteries in a human body [5]. The decomposition of stenosis in the artery causes the narrowing of the artery. Consequently, it affects the blood flows mechanism. One of the major causes of the deaths in the world is due to heart diseases, and the most commonly heard names among the same are ischemia, atherosclerosis, and angina pectoris [6]. Hence, mathematical model for modulation of non-linear waves in viscous fluid contained in stenosed elastic tube becomes one of the most significant study in order to determine the location of stenosis in the early stage.

In the past studies, many researchers have studied the wave propagation of fluid flow in an elastic tube such as [7], [8], [9], and [10] while wave modulation becomes less concerned. The study of wave modulation in the arteries is rather difficult to construct because the mathematical model involves complex solution. In 2017, Nagappan [11] studied on the non-linear wave modulation in a pre-stressed thin elastic tube where assuming that the artery is pre-stressed thin elastic, incompressible and isotropic tube whereas the blood is considered to be incompressible inviscid fluid. In this research, he obtained NLS equation. Ahmed et al. [12] investigated the modulation of non-linear wave in blood flow. The blood is considered to be incompressible inviscid fluid while the tube exhibits viscous and elastic behaviour. In this study, the NLS equation was obtained by using the reductive perturbation technique. Recently, Bi et al. [13] analyzed on the non-linear waves of blood flow in arterial vessels where considering the blood as an incompressible Newtonian fluid and the propagation of blood flow is observed. For their study, a new higher order non-linear Schrödinger equation is obtained to describe the blood flow in blood vessels.

Therefore, in this study, blood is treated as an incompressible viscous fluid and the artery is a pre-stressed, thin-walled and long elastic tube with a symmetrical stenosis. The method of reductive perturbation method is implemented in this study in order to investigate the modulation of non-linear waves in such composite medium. Later on, the dissipative NLS equation with variable coefficient was derived after solving the differential equations. Next, the progressive wave solution of the dissipative NLS equation with variable coefficient is determined. Results and discussion are also made for the variation of the radial displacement, as well as the fluid pressure function and the wave speed.

2. Equations of the tube and fluid

Since Rudinger [2] discovered that the blood behaved similarly to a viscous fluid, therefore in this study, the blood is treated as an incompressible viscous fluid. The equation of balance of linear momentum in the axial direction of an incompressible viscous fluid is given by [14]

$$\frac{\partial w^*}{\partial t^*}+w^*\frac{\partial w^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \mathrm{P}^*}{\partial z^*}-\frac{\mu_\upsilon}{\rho_f}\left[\frac{\partial^2 w^*}{\partial z^{*2}}-\frac{8w^*}{\left(r_0-f^*\left(z^*\right)+u^*\right)^2}\right]=0, \quad \text{Eq. 1}$$

where w^* denotes the mean of fluid speed, t^* is time, z^* represents a coordinate that located on axis when the changes of radius maintain its value, P^* is the mean of fluid pressure, μ_υ is the viscosity of the fluid, ρ_f is the mass density, $r_f=r_0-f^*\left(z^*\right)+u^*$ is the final radius after deformation occurred. The following equation is the mass conservation equation of an incompressible viscous fluid:

Eq. 2

where w^* denotes the mean of fluid speed, t^* is time. z^* represents a coordinate that located on axis when the changes of radius maintain its value, $f^*\left(z^*\right)$ is the function of a variable radius, υ defined

$$2\frac{\partial u^*}{\partial t^*}+2w^*\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)+\left[r_0-f^*\left(z^*\right)+u^*\right]\left(\frac{\partial w^*}{\partial z^*}\right)=0, \quad \text{Eq. 2}$$

as the viscosity for fluid flow, u^* is the function of displacement of the radius, while r_0 is the initial radius in the coordinate system.

Next, the equation of elastic tube in the radial direction could be written as follows [14]:

$$\mu R_0\frac{\partial}{\partial z^*}\left\{\frac{\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)}{\left[1+\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}}\frac{\partial\Sigma}{\partial\lambda_1}\right\}-\frac{\mu}{\lambda_z}\frac{\partial\Sigma}{\partial\lambda_2}+\frac{\left[1+\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}P^*}{H}\left(r_0-f^*+u^*\right)-\rho_0\frac{R_0}{\lambda_z}\frac{\partial^2u^*}{\partial t^{*2}}=0, \quad \text{Eq. 3}$$

where $\mu\Sigma$ depicts the membrane's strain energy density function while μ stands for the shear modulus of the tube material. In this equation, λ_z is the stretch ratio in the axial direction, H is the thickness of the non-deformed state, λ_1 and λ_2 denote the stretch ratios along the meridional and circumferential curves respectively while z^* is the axial coordinate after static deformation. Next, the R_0 represents the mean radius at the origin of the coordinate system while u^* denotes a dynamical radial displacement whereas P^* represents the inner pressure applied by the fluid. The r_0 in this equation is the deformed radius at the origin of the coordinate system while ρ_0 is the mass density of the tube and t^* is the time parameter. Figure 1 shows the geometry of the artery in various configurations.

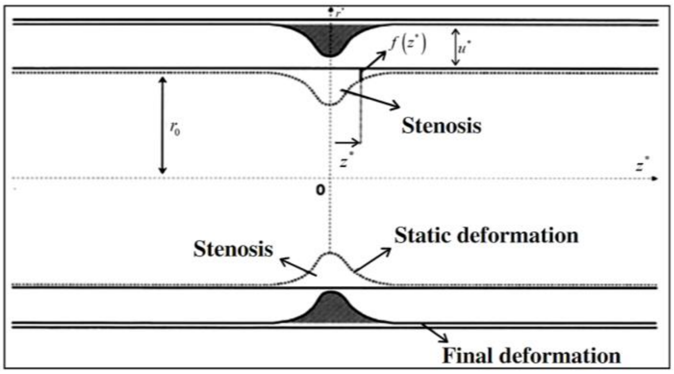


Figure 1: The geometry of the artery in various configurations.

The following non-dimensional quantities are introduced [14]:

$$t^* = \left(\frac{R_0}{c_0}\right)t, \quad z^* = R_0 z, \quad u^* = R_0 u, \quad f^* = R_0 f, \quad w^* = c_0 w, \quad \mu_\upsilon = c_0 R_0 \rho_f \bar{\upsilon},$$

$$P^* = \rho_f c_0^2 p, \quad r_0 = R_0 \lambda_\theta, \quad c_0^2 = \frac{\mu H}{\rho_f R_0}, \quad m = \frac{\rho_0 H}{\rho_f R_0}, \qquad \text{Eq. 4}$$

where $\lambda_\theta = \frac{r_0}{R_0}$ represents the initial stretch ratio, p is the fluid pressure, and c_0 denotes the Moens-Korteweg wave speed.

Introducing Eq. 4 into Eq. 1, Eq. 2, and Eq. 3 by applying chain rule results in

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} - \bar{\upsilon}\left[\frac{\partial^2 w}{\partial z^2} - \frac{8w}{\left(\lambda_\theta - f(z) + u\right)^2}\right] = 0,$$

$$2\frac{\partial u}{\partial t} + 2w\left(-f' + \frac{\partial u}{\partial z}\right) + \left[\lambda_\theta - f(z) + u\right]\frac{\partial w}{\partial z} = 0,$$

$$p = \frac{m}{\lambda_z\left(\lambda_\theta - f(z) + u\right)}\left(\frac{\partial^2 u}{\partial t^2}\right) + \frac{1}{\lambda_z\left(\lambda_\theta - f(z) + u\right)}\frac{\partial \Sigma}{\partial \lambda_2}$$

$$-\frac{1}{\left(\lambda_\theta - f(z) + u\right)}\frac{\partial}{\partial z}\left\{\frac{-f' + \frac{\partial u}{\partial z}}{\left[1 + \left(-f' + \frac{\partial u}{\partial z}\right)^2\right]^{\frac{1}{2}}}\frac{\partial \Sigma}{\partial \lambda_1}\right\} + \frac{4\bar{\nu}\left(-f' + \frac{\partial u}{\partial z}\right)w}{\left(\lambda_\theta - f(z) + u\right)}. \qquad \text{Eq. 5}$$

3. Non-linear Wave Modulation

In this section, the method of reductive perturbation is applied. Firstly, the following type of stretched coordinates are introduced [14]:

$$\xi = \varepsilon\left(z - \lambda t\right), \quad \tau = \varepsilon^2 z, \qquad \text{Eq. 6}$$

where ξ denotes the wave profile, τ represents the space, ε represents the small parameter that measures the weakness of non-linearity, and λ indicates the constant to be determined from the solution.

Since this study is to investigate the effect of stenosis, hence $f(z)$ should be in first-order, $O(\varepsilon)$ whereby $\hat{h}(\tau) = h(\tau)$ [14]. The differential relations are introduced in the following form [14]:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \varepsilon\lambda\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \varepsilon\frac{\partial}{\partial \xi} + \varepsilon^2\frac{\partial}{\partial \tau}. \qquad \text{Eq. 7}$$

For the long wave limit approximation, the function $h(\tau)$ and the field quantities u, w, and p are assumed that it can be written as asymptotic series in the following form [14]:

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \ldots, \quad w = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \ldots,$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \ldots,\ h(\tau) = \varepsilon^2 h_1(\tau) + \varepsilon^3 h_2(\tau) + \ldots \qquad \text{Eq. 8}$$

Introducing the stretched coordinates Eq. 6, differential relations Eq. 7 and asymptotic series Eq. 8 into Eq. 5, the various order equations obtained are shown below:

$O(\varepsilon)$ equation:

$$\frac{\partial w_1}{\partial t} + \frac{\partial p_1}{\partial z} = 0, \qquad \frac{\partial u_1}{\partial t} + \lambda_\theta \frac{\partial w_1}{\partial z} = 0, \qquad p_1 = \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^2 u_1}{\partial t^2} - \alpha_0 \frac{\partial^2 u_1}{\partial z^2} + \beta_1 u_1. \qquad \text{Eq. 9}$$

$O(\varepsilon^2)$ equation:

$$\frac{\partial w_2}{\partial t} + \frac{\partial p_2}{\partial z} - \lambda \frac{\partial w_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} + w_1 \frac{\partial w_1}{\partial z} = 0, \qquad 2\frac{\partial u_2}{\partial t} + \lambda_\theta \frac{\partial w_2}{\partial z} - 2\lambda \frac{\partial u_1}{\partial \xi} + \lambda_\theta \frac{\partial w_1}{\partial \xi} + u_1 \frac{\partial w_1}{\partial z} + 2 w_1 \frac{\partial u_1}{\partial z} = 0,$$

$$p_2 = \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^2 u_2}{\partial t^2} - \alpha_0 \frac{\partial^2 u_2}{\partial z^2} + \beta_1 (u_2 - h_1) - \frac{2m\lambda}{\lambda_\theta \lambda_z}\frac{\partial^2 u_1}{\partial \xi \partial t} - 2\alpha_0 \frac{\partial^2 u_1}{\partial \xi \partial z} - \frac{m}{\lambda_\theta^2 \lambda_z} u_1 \frac{\partial^2 u_1}{\partial t^2} - \alpha_1 \left(\frac{\partial u_1}{\partial z}\right)^2$$
$$-\left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right) u_1 \frac{\partial^2 u_1}{\partial z^2} + \beta_2 u_1^2. \qquad \text{Eq. 10}$$

$O(\varepsilon^3)$ equation:

$$\frac{\partial w_3}{\partial t} + \frac{\partial p_3}{\partial z} - \lambda \frac{\partial w_2}{\partial \xi} + \frac{\partial p_2}{\partial \xi} + \frac{\partial p_1}{\partial \tau} + w_1 \frac{\partial w_2}{\partial z} + w_2 \frac{\partial w_1}{\partial z} + w_1 \frac{\partial w_1}{\partial \xi} - \upsilon \left(\frac{\partial^2 w_1}{\partial z^2} - \frac{8 w_1}{\lambda_\theta^2}\right) = 0,$$

$$2\frac{\partial u_3}{\partial t} + \lambda_\theta \frac{\partial w_3}{\partial z} - 2\lambda \frac{\partial u_2}{\partial \xi} + \lambda_\theta \frac{\partial w_2}{\partial \xi} + 2 w_1 \left(\frac{\partial u_1}{\partial \xi} + \frac{\partial u_2}{\partial z}\right) + 2 w_2 \frac{\partial u_1}{\partial z} + \lambda_\theta \frac{\partial w_1}{\partial \tau} + u_1 \left(\frac{\partial w_2}{\partial z} + \frac{\partial w_1}{\partial \xi}\right) + \left[u_2 - h_1(\tau)\right]\frac{\partial w_1}{\partial z} = 0,$$

$$p_3 = \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^3 u_3}{\partial t^3} - \alpha_0 \frac{\partial^2 u_3}{\partial z^2} - \frac{2m\lambda}{\lambda_\theta \lambda_z}\frac{\partial^2 u_2}{\partial \xi \partial t} - 2\alpha_0 \frac{\partial^2 u_2}{\partial \xi \partial z} - \alpha_0 \left(\frac{\partial^2 u_1}{\partial \xi^2} + 2\frac{\partial^2 u_1}{\partial z \partial \tau}\right) + \frac{m\lambda^2}{\lambda_\theta^2 \lambda_z}\frac{\partial^2 u_1}{\partial \xi^2} + \beta_1 (u_3 - h_2)$$
$$-\frac{m}{\lambda_\theta^2 \lambda_z} u_1 \left(\frac{\partial^2 u_2}{\partial t^2} - 2\lambda \frac{\partial^2 u_1}{\partial \xi \partial t}\right) - \frac{m}{\lambda_\theta^2 \lambda_z}(u_2 - h_1)\frac{\partial^2 u_1}{\partial t^2} - 2\alpha_1 \frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi}\right) - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right) u_1$$
$$\times \left(\frac{\partial^2 u_2}{\partial z^2} + 2\frac{\partial^2 u_1}{\partial z \partial \xi}\right) - 2\alpha_1 \frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi}\right) - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right) u_1 \left(\frac{\partial^2 u_2}{\partial z^2} + 2\frac{\partial^2 u_1}{\partial z \partial \xi}\right) + 2\beta_2 u_1 (u_2 - h_1)$$
$$+\frac{m}{\lambda_\theta^3 \lambda_z} u_1^2 \frac{\partial^2 u_1}{\partial t^2} - \left(\alpha_2 - \frac{\alpha_1}{\lambda_\theta}\right) u_1 \left(\frac{\partial u_1}{\partial z}\right)^2 - \left(\alpha_2 - \frac{2\alpha_1}{\lambda_\theta} + \frac{\alpha_0}{\lambda_\theta^2}\right) u_1^2 \frac{\partial^2 u_1}{\partial z^2} - 3\left(\gamma_1 - \frac{\alpha_0}{2}\right)\left(\frac{\partial u_1}{\partial z}\right)^2 \left(\frac{\partial^2 u_1}{\partial z^2}\right) + \beta_3 u_1^3. \qquad \text{Eq. 11}$$

Here the coefficients of α_0, α_1, α_2, β_0, β_1, β_2, β_3, and γ_1 are defined by [14]:

$$\alpha_0 = \frac{1}{\lambda_\theta}\frac{\partial \Sigma}{\partial \lambda_z}, \quad \alpha_1 = \frac{1}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_\theta \lambda_z}, \quad \alpha_2 = \frac{1}{2\lambda_\theta}\frac{\partial^3 \Sigma}{\partial \lambda_\theta^2 \lambda_z}, \quad \beta_0 = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial \Sigma}{\partial \lambda_\theta}, \quad \beta_1 = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial^2 \Sigma}{\partial \lambda_\theta^2} - \frac{\beta_0}{\lambda_\theta}, \quad \beta_2 = \frac{1}{2\lambda_\theta \lambda_z}\frac{\partial^3 \Sigma}{\partial \lambda_\theta^3} - \frac{\beta_1}{\lambda_\theta},$$
$$\beta_3 = \frac{1}{6\lambda_\theta \lambda_z}\frac{\partial^4 \Sigma}{\partial \lambda_\theta^4} - \frac{\beta_2}{\lambda_\theta}, \quad \gamma_1 = \frac{\lambda_z}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_z^2}. \qquad \text{Eq. 12}$$

The Solution of $O(\varepsilon)$ equations

For the first-order Eq. 9, the following type of solution are explored [14]:

$$u_1 = \left(U_1 e^{i\theta} + c.c\right), \quad w_1 = \left(W_1 e^{i\theta} + c.c\right), \quad p_1 = \left(-\frac{m\omega^2}{\lambda_\theta \lambda_z} + \alpha_0 k^2 + \beta_1\right) U_1 e^{i\theta} + c.c, \qquad \text{Eq. 13}$$

where U_1 and W_1 are the unknown functions of the slow variables (ξ,τ), $\theta=\omega t-kz$ indicates the phasor and $c.c$ denotes the complex conjugate of the corresponding expressions. Here, ω represents the angular frequency whereas k represents the wave number.
Applying Eq. 13 into Eq. 9, it yields

$$U_1=U(\xi,\tau),\quad W_1=\frac{2\omega}{\lambda_\theta k}U. \qquad \text{Eq. 14}$$

Solving Eq. 9 results in

$$\omega^2=\frac{\lambda_\theta\lambda_z k^2\left(\alpha_0 k^2+\beta_1\right)}{2\lambda_z+mk^2}. \qquad \text{Eq. 15}$$

The Solution of $O(\varepsilon^2)$ equations

Introducing the following expression to obtain the solutions for $O(\varepsilon^2)$ [14]:

$$u_2=U_2^{(0)}+\left(\sum_{l=1}^{2}U_2^{(l)}e^{il\theta}+c.c\right),\quad w_2=W_2^{(0)}+\sum_{l=1}^{2}W_2^{(l)}e^{il\theta}+c.c,\quad p_2=P_2^{(0)}+\sum_{l=1}^{2}P_2^{(l)}e^{il\theta}+c.c. \qquad \text{Eq. 16}$$

Introducing the solutions Eq. 14 and Eq. 15 into Eq. 10 gives

$$p_2=\frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_2}{\partial t^2}-\alpha_0\frac{\partial^2 u_2}{\partial z^2}+\beta_1\left(u_2-h_1\right)+2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\alpha_1 k^2-\frac{\alpha_0 k^2}{\lambda_\theta}+\beta_2\right)|U|^2$$
$$+2i\left(\alpha_0 k-\frac{m\omega\lambda}{\lambda_\theta\lambda_z}\right)\frac{\partial U}{\partial\xi}e^{i\theta}+\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+3\alpha_1 k^2-\frac{\alpha_0 k^2}{\lambda_\theta}+\beta_2\right)U^2e^{2i\theta}+c.c,$$

$$2\frac{\partial u_2}{\partial t}+\lambda_\theta\frac{\partial w_2}{\partial z}+2\left(\frac{\omega}{k}-\lambda\right)\frac{\partial U}{\partial\xi}e^{i\theta}-6i\frac{\omega}{\lambda_\theta}U^2e^{2i\theta}+c.c=0,$$

$$\frac{\partial w_2}{\partial t}+\frac{\partial p_2}{\partial z}+\left(-2\frac{\lambda\omega}{\lambda_\theta k}-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0 k^2+\beta_1\right)\frac{\partial U}{\partial\xi}e^{i\theta}-4i\frac{\omega^2}{\lambda_\theta^2 k}U^2e^{2i\theta}+c.c=0, \qquad \text{Eq. 17}$$

where $|U|^2=UU^*$, U^* represents the complex conjugate of U.
Solving Eq. 17 by applying Eq. 16 yields:

$$P_2^{(1)}=\left(-\frac{m}{\lambda_\theta\lambda_z}\omega^2+\alpha_0 k^2+\beta_1\right)U_2^{(1)}+2i\left(\alpha_0 k-\frac{m\omega\lambda}{\lambda_\theta\lambda_z}\right)\frac{\partial U}{\partial\xi}, \qquad \text{Eq. 18}$$

$$W_2^{(1)}=\frac{2i}{k\lambda_\theta}\left(\lambda-\frac{\omega}{k}\right)\frac{\partial U}{\partial\xi}, \qquad \text{Eq. 19}$$

the group velocity, $$\lambda=\frac{\lambda_z\left[\lambda_\theta k^4\alpha_0+2\omega^2\right]}{\omega k\left(2+mk^2\right)}, \qquad \text{Eq. 20}$$

$$W_2^{(2)}=\frac{2\omega}{\lambda_\theta k}U_2^{(2)}-\frac{3\omega}{\lambda_\theta^2 k}U^2$$, and Eq. 21

$U_2^{(2)}=\Phi_0 U^2$, where $$\Phi_0=\frac{\dfrac{3\omega^2}{\lambda_\theta}+k^2\beta_1+3\alpha_1 k^4\lambda_\theta+\lambda_\theta\beta_2 k^2}{3\left(\beta_1\lambda_\theta k^2-2\omega^2\right)}. \qquad \text{Eq. 22}$$

The Solution of $O\left(\varepsilon^3\right)$ equations

To obtain the solutions for $O\left(\varepsilon^3\right)$, firstly, the following expressions are introduced [14]:

$$u_3 = U_3^{(0)} + \left(\sum_{l=1}^{3} U_3^{(l)} e^{il\theta} + c.c\right), \quad w_3 = W_3^{(0)} + \sum_{l=1}^{3} W_3^{(l)} e^{il\theta} + c.c, \quad p_3 = P_3^{(0)} + \sum_{l=1}^{3} P_3^{(l)} e^{il\theta} + c.c \qquad \text{. Eq. 23}$$

Apply the harmonic wave solutions Eq. 23 into $O\left(\varepsilon^3\right)$ Eq. 11 yields the following dissipative NLS equation with variable coefficients [14]:

$$i\frac{\partial U}{\partial \xi} + \mu_1 \frac{\partial^2 U}{\partial \xi^2} + \mu_2 \left|U\right|^2 U - \mu_3 h_1(\tau) U + i\mu_4 U = 0, \qquad \text{Eq. 24}$$

whereby U is the unknown function and the coefficients of μ_1, μ_2, μ_3 and μ_4 are defined by

$$\mu = \frac{2\omega^2}{k} + 3\alpha_0\lambda_\theta k^3 - \frac{m\omega^2 k}{\lambda_z} + \lambda_\theta \beta_1 k, \quad \mu_1 = \mu^{(-1)}\left[-\frac{4\lambda\omega}{k} + \frac{2\omega^2}{k^2} + 2\lambda^2 + \frac{m\lambda^2 k^2}{\lambda_z} - 3\alpha_0\lambda_\theta k^2 + \frac{2m\omega\lambda k}{\lambda_z}\right],$$

$$\mu_2 = \mu^{(-1)}\left\{\left[\frac{10\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{5m\omega^2}{\lambda_\theta^2\lambda_z} + 6\alpha_1 k^2 - \frac{5\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_0 + \left[\frac{8\lambda k}{\omega\lambda_\theta} + \frac{2\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta\lambda_z} + 2\alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_1\right.$$

$$\left. + \lambda_\theta k^2\left[-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z} + 2\alpha_2 k^2 - \frac{5\alpha_1 k^2}{\lambda_\theta} + \frac{3\alpha_0 k^2}{\lambda_\theta^2} + 3\left(\gamma_1 - \frac{\alpha_0}{2}\right)k^4 + 3\beta_3\right] - \frac{30\omega^2}{\lambda_\theta^2}\right\},$$

$$\mu_3 = \mu^{(-1)}\left\{\left[\frac{8\omega\lambda k}{\lambda_\theta} + \frac{2\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta\lambda_z} + 2\alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_2 + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta\lambda_z} + 2\alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right\},$$

$$\mu_4 = \mu^{(-1)}\left[2\upsilon\omega\left(k^2 + \frac{8}{\lambda_\theta^2}\right)\right]. \qquad \text{Eq. 25}$$

The k is known as the number of wave, ω represents the angular frequency while υ is the fluid viscosity.

The following changing of variable is applied by [14]:

$$U = V(\xi,\tau)\, e^{-i\mu_3\int_0^\tau h_1(s)ds - \mu_4\tau}, \qquad \text{Eq. 26}$$

which can reduces Eq. 24 to the following conventional NLS equations [14]:

$$i\frac{\partial V}{\partial \tau} + \mu_1 \frac{\partial^2 V}{\partial \xi^2} + \mu_2 \left|V\right|^2 V = 0. \qquad \text{Eq. 27}$$

4. Progressive Wave Solution

In the previous section, the coefficient μ_4 describes the dissipation resulting from the viscosity of the fluid whereas the coefficients μ_1, μ_2, and μ_3 contribute the variable radius of the tube. The progressive wave solution is applied into Eq. 27 of the following term:

$$V(\xi,\tau) = F(\zeta) e^{i(K\xi - \Omega\tau)}, \qquad \zeta = \xi - c\tau. \qquad \text{Eq. 28}$$

Substituting Eq. 28 into Eq. 27 will results in

$$\mu_1 \frac{\partial^2 F}{\partial \zeta^2} + i\left(2\mu_1 K - c\right)\frac{\partial F}{\partial \zeta} + \left(\Omega - \mu_1 K^2\right)F + \mu_2 F^3 = 0,$$ Eq. 29

whereby the product of $\mu_1\mu_2 < 0$ and $\Omega = \mu_1 K^2 + \mu_3 h_1(\tau) - \mu_2 a^2 e^{-\frac{4}{3}\mu_4 \tau}$. Here, a represents the amplitude of the wave. In order to eliminate the $\frac{\partial F}{\partial \zeta}$ term, let $c = 2\mu_1 K$, then Eq. 29 yields

$$\mu_1 \frac{\partial^2 F}{\partial \zeta^2} - \left(\mu_3 h_1(\tau) - \mu_2 a^2 e^{-\frac{4}{3}\mu_4 \tau}\right)F + \mu_2 F^3 = 0.$$ Eq. 30

By solving the Eq. 30, the Eq. 28 can be expressed as

$$V(\xi,\tau) = a \tanh\left[\sqrt{-\frac{\mu_2}{2\mu_1}}\left(\xi - c\tau\right)\right] e^{i(K\xi - \Omega\tau)}.$$ Eq. 31

Applying Eq. 31 into Eq. 26 gives the solution of the dissipative NLS equation with variable coefficient Eq. 24 as

$$U(\xi,\tau) = a \tanh\left[\sqrt{-\frac{\mu_2}{2\mu_1}}\left(\xi - c\tau\right)\right] e^{i\left(K\xi - \Omega\tau - \mu_3 \int_0^\tau h_1(s)ds\right) - \mu_4 \tau}.$$ Eq. 32

By setting $i\left(K\xi - \Omega\tau - \mu_3 \int_0^\tau h_1(s)ds\right) - \mu_4 \tau = 0$, it gives

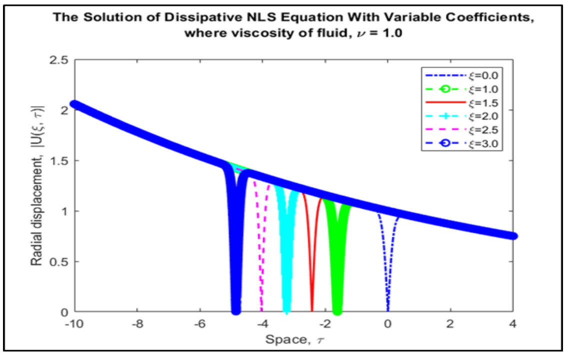
$$\xi = \frac{1}{K}\left(\Omega\tau + \mu_3 H_1(\tau) + \frac{1}{i}\mu_4 \tau\right).$$ Eq. 33

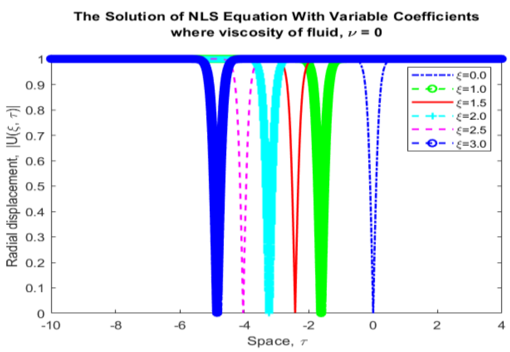
Solving the above equation, it yields to the result for the wave speed of the NLS equation with variable coefficients Eq. 24

$$\upsilon_p = \frac{K}{\left(\Omega + \mu_3 h_1(\tau) - i\mu_4\right)}.$$ Eq. 34

5. Results and Discussion

In this study, the graphical results for radial displacement, the effect of viscous in fluid, fluid pressure, and wave speed are illustrated by using MATLAB software. The numerical value of α was found to be 1.948 [15]. By using the values of $\alpha = 1.948$, $\lambda_z = \lambda_\theta = 1.6$ where λ_z is axial stretch and λ_θ represents the stretch ratio in the circumferential direction while $m = 0.1$, $\upsilon = 1$, $k = 2$ and $K = 2$, it gives the results $\alpha_0 = 78.6924$, $\alpha_1 = 233.7666$, $\alpha_2 = 1563.4837$, $\beta_0 = 49.1827$, $\beta_1 = 296.1049$, $\beta_2 = 991.4958$, $\beta_3 = 2394.6580$, $\gamma_1 = 418.3605$, $\omega = 41.6845$, $\lambda = 29.2660$, $\Phi_0 =$ -6.0631, $\Phi_1 = 7.2986$, $\Phi_2 = 0.3823$, $\mu_1 =$ -0.1548, $\mu_2 = 27.7531$, $\mu_3 = 7.3572$, and $\mu_4 = 0.1082$.

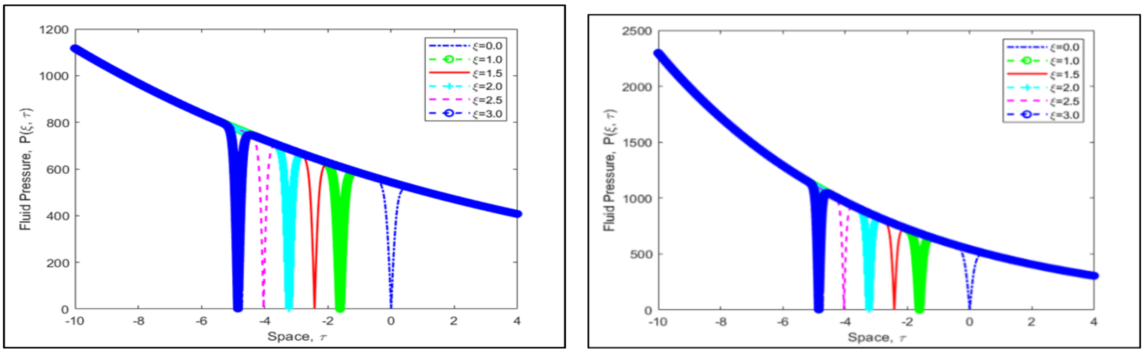




(a) (b)

Figure 2: The solution of dissipative NLS equation with variable coefficients versus space, τ at fluid viscosity, $\upsilon = 1.0$ for (a) and the solution of NLS equation for (b).

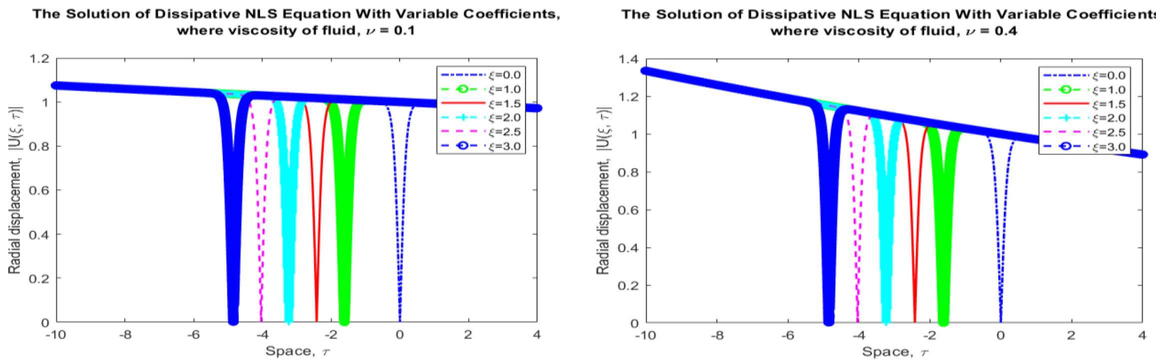
Figure 2 illustrates the comparison between the viscous fluid (a) and the inviscid fluid (b) flows in the thin walled elastic tube with stenosis. As can been seen in these figures, the viscous effect of fluid influenced the wave amplitude. When viscous fluid flows in the stenosed elastic tube, the wave propagates to the left with increasing amplitude due to the effect of viscosity. On the other sides, when the inviscid fluid flows in the stenosed elastic tube, the downward bell-shaped wave with the amplitude of unity travels to the left.



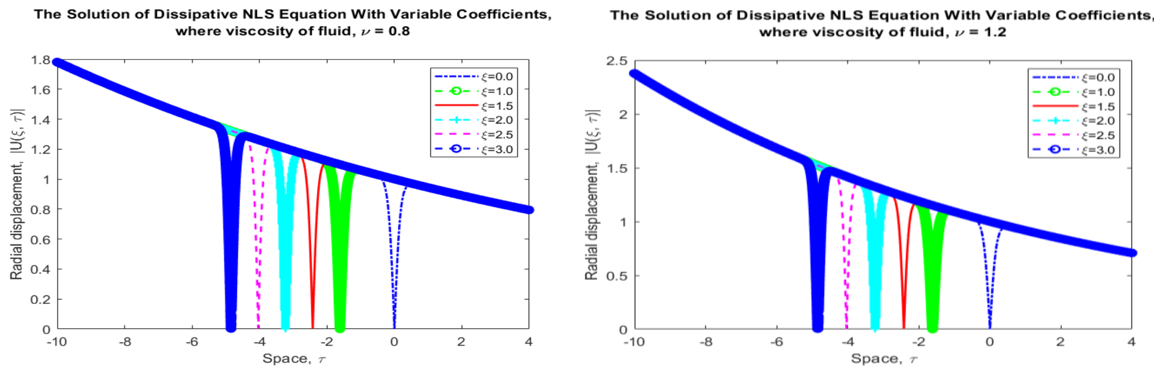
(a) (b)

Figure 3: The fluid pressure versus space, τ at different fluid viscosity, (a) $\upsilon = 1.0$ and (b) $\upsilon = 2.0$.

Figure 3 presents the behaviour of the fluid pressure for the two different fluid viscosity. It is worth noting that the resistance to flow is greatly enhanced even for a slight increment of the viscous effect of fluid. In other words, the fluid pressure are observed to be increasing with the increase in the fluid viscosity, which means that the blood required higher pressure to passes through the stenosis when the viscous effect of fluid increases.



(a) (b)



(c) (d)

Figure 4: The solution of dissipative NLS equation with variable coefficients versus space, τ with various viscosity of fluid, (a) $\upsilon = 0.1$, (b) $\upsilon = 0.4$, (c) $\upsilon = 0.8$ and $\upsilon = 1.2$.

Figure 4 exhibits the variation of the radial displacement, $U(\xi,\tau)$ with space, τ when viscosity of fluid is $\upsilon = 0.1,\ 0.4,\ 0.8,\ \text{and}\ 1.2$. As the fluid viscosity increases, the downward bell-shaped wave travels to the left with increasing amplitude. Thus, the severity of the viscous effect of blood affects the blood flow in the stenosed artery significantly. The higher of the viscous effect of blood, the higher the severity, the more the radial displacement is increased due to resistance in flow increase.

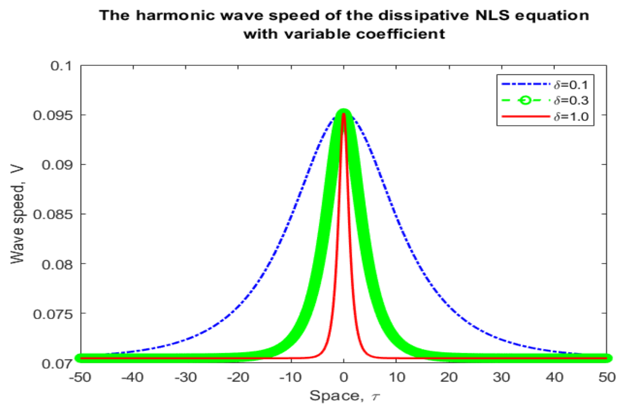


Figure 5: The wave speed, V of the NLS equation with variable coefficients at viscosity of fluid, $\upsilon = 1.0$.

In Figure 5, δ determines the sharpness of stenosis function, $f(\tau) = \text{sech}(\delta\tau)$. Figure 4 indicates that blood velocity increases as it passes through the stenotic region. The wave speed takes it maximum value at the center of stenosis and it gets smaller as it goes away from the center of stenosis approaching a constant value. The severity of the stenosis affects the wave speed significantly. That is, the more the severity, the bell-shaped curve of wave speed appears to be more sharply.

6. Conclusion

In this study, a mathematical model of wave modulation for blood flow in a thin walled stenosed elastic artery has been developed. The dimension equations of tube and fluid are transformed into dimensionless equations by substituting the dimensionless quantities. Throughout this study, the RPM is implemented in the dimensionless equations of tube and fluid to get various orders of the differential equations. After that, the dissipative NLS equation with variable coefficients is obtained after solved the differential equations. Later on, the progressive wave solution is applied in the NLS equation with variable coefficients to achieve the analytical solution. The graphical outputs were illustrated by using MATLAB software in order to discuss the consequences of analytical solution on radial displacement, the fluid viscous effect, and wave speed.

It is observed that the radial displacement and fluid pressure increase when the fluid viscosity increases. The modulus of the radial displacement and fluid pressure show a downward bell shaped wave solution. The waves propagate to the left with increasing amplitude. On the other hand, it is found that the wave speed reaches to its maximum value at the center of the stenosis and becomes smaller and smaller as it go away from the stenosis. This result seems to be reasonable from the physical view point.

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