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A Non-Linear Shooting Method For Two-Point Boundary Value Problems

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Abstract: In this research, non-linear two-point boundary value problems were solved numerically using shooting method which involved shooting based on secant method and Newton Raphson's method. The boundary value problems were first reduced to the solution of initial value problem and solved by using Fourth Order Runge-Kutta method with the aid of MATLAB. Shooting method then applied when the approximate solution did not converge within the provided error tolerance. After that, the numerical results obtained then being compared with the exact solution and Finite Difference Method to check the efficiency and accuracy of using shooting method in solving the non-linear two-point boundary value problems.

Keywords: Non-Linear Boundary Value Problems, Shooting Method, Secant Method, Newton Raphson's Method

1. Introduction

Boundary value problems is a system of ordinary differential equation with more than one point of specified solution and derivatives values usually two points which are initial value and terminal value resulting in two-point boundary value problem [1]. Shooting method as an algorithm that aims to define desirable initial conditions for a specific initial value problem providing a solution to the original boundary value problem. This method visualizes the reduction of the differential equations of boundary value problems to the solution of the initial value problem (IVP), presumed that the initial value given was for an initial value problem of ordinary differential equations [2]. The initial value then calculated using methods such as Fourth Order Runge-Kutta (RK4) and other suitable methods before proceeding with the shooting of boundary values [3]. From a previous study, Ahsan and Farrukh developed a new technique where both shooting method and interpolation were used to increase the accuracy in solving both linear and non-linear boundary value problems [4]. Another study by Filipov solved a two-point boundary value problem by using shooting method where integration was used to acquire the solution of initial value problem [5]. For this research, we focused on the solution of two-point boundary value problem in the case of non-linear by the implementation of shooting technique with secant method and Newton Raphson's method. The problems first insisted on the reducing of boundary value problem into

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an initial value problem before shooting the first value of initial guess to solve it. Shooting a new value for initial guess continues through a series of trial and error in order to ensure the possibility of getting a final value as close as the real boundary value.

2. Methodology

To apply shooting method, the boundary value problems expressed in (2.1) will be first reduced to the solution of initial value problems stated in (2.2) where

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta, \tag{2.1}$$

$$w'' = f(x, w, w'), \qquad a \le x \le b, \qquad w(a) = \alpha, \qquad w'(a) = v_k, \tag{2.2}$$

in which solution to the initial value problem (2.2) is referring to $w(x, v_k)$ with $v = v_k$. As we let our initial elevation as $v = v_0$, the final solution of IVP was determined by considering that initial slope start from the (a, α) and along the curve

$$w'' = f(x, w, w'), \; a \le x \le b, \; w(a) = \alpha, \quad w'(a) = v_0. \tag{2.3}$$

A new elevation, say v_1, will be utilize if the previous $w(b, v_k)$ is not close enough to the value of β. Through series of trial and error, the process of choosing a suitable elevation of v_k will continue until $w(b, v_k)$ is close to "approaching" β and converges within the given error tolerance of

$$|w(b, v_k) - \beta| = |\phi(v_k) - \beta| \le \varepsilon = 0.005. \tag{2.4}$$

2.1 Shooting Method based on Secant Method

In order to apply shooting based on secant, we first solved the problem stated in (2.3) with two different initial guess say v_0 and v_1. Since the solution for (2.3) are depend on v, we indicate that

$$w(b) = \phi(v_k), \tag{2.5}$$

where (2.5) shows the non-linear function of $w(b)$ depends on v. Say both $\phi(v_k)$ for v_0 and v_1 did not fulfil the condition of (2.4), then we need to shoot the new elevation of v_k using secant formula where

$$v_k = v_{k-1} - \frac{v_{k-1} - v_{k-2}}{\phi(v_{k-1}) - \phi(v_{k-2})} \cdot [|\beta - \phi(v_{k-1})|], \quad \text{where} \quad k = 2,3,4,... \; . \tag{2.6}$$

the new v_k is used to solved (2.3) by using Fourth-Order Runge-Kutta method to obtain the value of $\phi(v_k)$. The process of choosing new v_k in (2.6) is used until the condition in (2.4) is fulfilled. On occasion that $\phi(v_k)$ fulfil the condition in (2.4), the boundary value problems will be considered solved.

2.2 Shooting Method based on Newton Raphson's Method

The process of shooting initial guesses of v_k using Newton Raphson's can be generating by

$$v_k = v_{k-1} - \frac{(w(b, v_{k-1}) - \beta)}{\left(\frac{dw}{dv}\right)(b, v_{k-1})}, \quad \text{where} \quad \left(\frac{dw}{dv}\right)(b, v_{k-1}) = \frac{dw}{dw}(b, v_{k-1}), \tag{2.7}$$

in which the need of understanding $\left(\frac{dw}{dv}\right)(b, v_{k-1})$ is necessary. In contrast to the difficulty of solving $\left(\frac{dw}{dv}\right)(b, v_{k-1})$, say that the equation in (2.2) was modified where we highlight that the solution relies on both x and v,

$$w''(x,v) = f(x, w(x,v), w'(x,v)), \quad a \le x \le b, \quad w(a,v) = \alpha, \quad w'(a,v) = v, \tag{2.8}$$

which implying differentiation with respect to x. By partial derivative of (2.8) with respect to v, we can resolve $\left(\frac{dw}{dv}\right)(b,v)$ when $v = v_{k-1}$ which implicates

$$\frac{\partial w''}{\partial v}(x,v) = \frac{\partial f}{\partial w}(x, w(x,v), w'(x,v))\frac{\partial w}{\partial v}(x,v) + \frac{\partial f}{\partial w'}(x, w(x,v), w'(x,v))\frac{\partial w'}{\partial v}(x,v), \tag{2.9}$$

with given initial conditions of

$$\frac{\partial w}{\partial v}(a,v) = 0 \quad \text{and} \quad \frac{\partial w'}{\partial v}(a,v) = 1.$$

If $\left(\frac{\partial w}{\partial v}\right)(x,v)$ can be rationalize as $z(x,v)$ and the order of differentiation for x and v can be inverted, the initial value problem for (2.9) will be in the form of

$$z'' = \frac{\partial f}{\partial w}(x, w, w')z + \frac{\partial f}{\partial w'}(x, w, w')z', \quad a \le x \le b, \quad z(a) = 0, \quad z'(a) = 1, \tag{2.10}$$

where both (2.2) and (2.10) must be solved simultaneously. From (2.7),

$$v_k = v_{k-1} - \frac{v(b, v_{k-1}) - \beta}{z(b, v_{k-1})}. \tag{2.11}$$

We can say that the sequence of v_k continues until $w(b, v_k)$ fulfil the condition in (2.4).

3. Results and Discussion

We solved two problem of non-linear two-point boundary value problems from a study by [1] using shooting method and compared the result with the solution of problems solved by Finite Difference Method (FDM). The solution for the problems were obtained by using of MATLAB and Maple 2015.

3.1 Problem 1

Consider the problem as follows

$$y''(x) = 2y^3 - 6y - 2x^3, \quad 1 \le x \le 2, \quad y(1) = 2, \quad \text{and} \quad y(2) = \frac{5}{2}, \tag{3.1}$$

with exact solution given by

$$y(x) = x + \frac{1}{x}. \tag{3.2}$$

a) Shooting by secant method

To solve using shooting by secant, we first change (3.1) into the form of (2.2) where

$$w''(x) = 2w^3 - 6w - 2x^3, \quad 1 \le x \le 2, \quad w(1) = 2, \quad w'(1) = v_k. \tag{3.3}$$

To apply Runge-Kutta method in solving the problem, we then expressed (3.3) in the form of first-order differential equation, then

$$u = w', \quad u' = 2w^3 - 6w - 2x^3, \quad w(1) = 2, \quad u(1) = v_k. \tag{3.4}$$

We first choose two value of v_k say that $v_0 = 0.02$ and $v_1 = 0.03$. Substituting both value into (3.4), we solved the problem simultaneously with step size of $h = 0.1$. Then, the new v_k can be find by using the formula stated in (2.6) and it continues until $\phi(v_k)$ fulfil the condition of (2.4). Using shooting with secant, problem 1 needs four iteration for approximate solution to converge within the provided error tolerance.

Table 1: Approximate solution of Problem 1 with different values of v_k (Secant)

$w(i)$	$v_0 = 0.02$	$v_1 = 0.03$	$v_2 = 0.0013$	$v_3 = 0.0001$	Exact Solution
1.0	2.0000	2.0000	2.0000	2.0000	2.0000
1.1	2.0112	2.0122	2.0092	2.0091	2.0091
1.2	2.0378	2.0401	2.0336	2.0334	2.0333
1.3	2.0771	2.0810	2.0697	2.0693	2.0692
1.4	2.1270	2.1334	2.1151	2.1143	2.1143
1.5	2.1870	2.1972	2.1680	2.1668	2.1667
1.6	2.2576	2.2740	2.2271	2.2251	2.2500
1.7	2.3412	2.3680	2.2916	2.2884	2.2882
1.8	2.4431	2.4879	2.3610	2.3559	2.3556
1.9	2.5742	2.6513	2.4355	2.4268	2.4263
2.0	2.7567	2.8941	2.5156	2.5009	2.5000

b) Shooting by Newton Raphson's method

We first expressed (2.10) for problem 1 in which

$$z'' = (6w^2 - 6)z, \quad 1 \le x \le 2, \quad z(1) = 0, \quad z'(1) = 1, \tag{3.5}$$

and changed (3.5) into the form of first-order differential equation where

$$t = z', \quad t' = (6w^2 - 6)z, \quad z(1) = 0, \quad t(1) = 1. \tag{3.6}$$

Solving both (3.3) and (3.6) simultaneously with $v_0 = 0.02$ to obtain the value of $w(b, v_0)$, and then proceed with the process of choosing new v_k using formula in (2.11). Through shooting with Newton's, three iterations were needed in order for the $w(b, v_k)$ to converge within the provided tolerance error.

Table 2: Approximate solution of Problem 1 with different values of v_k (Newton's)

$w(i)$	$v_0 = 0.02$	$v_1 = 0.0009$	$v_2 = 0.00003$	Exact Solution
1.0	2.0000	2.0000	2.0000	2.0000
1.1	2.0112	2.0092	2.0091	2.0091
1.2	2.0378	2.0335	2.0334	2.0333
1.3	2.0771	2.0696	2.0692	2.0692
1.4	2.1270	2.1149	2.1143	2.1143
1.5	2.1870	2.1676	2.1667	2.1667
1.6	2.2576	2.2264	2.2250	2.2500
1.7	2.3412	2.2905	2.2882	2.2882

1.8	2.4431	2.3593	2.3556	2.3556
1.9	2.5742	2.4326	2.4263	2.4263
2.0	2.7567	2.5107	2.5000	2.5000

c) Solution of Problem 1 using Finite Difference Method (FDM)

Table 3 shows the solution of Problem 1 using FDM.

Table 3: Numerical solution of Problem 1 using FDM

$w(i)$	FDM Solution	Exact Solution
1.0	2.0000	2.0000
1.1	2.0092	2.0091
1.2	2.0335	2.0333
1.3	2.0694	2.0692
1.4	2.1144	2.1143
1.5	2.1668	2.1667
1.6	2.2251	2.2500
1.7	2.2883	2.2882
1.8	2.3556	2.3556
1.9	2.4263	2.4263
2.0	2.5000	2.5000

3.2 Problem 2

Consider the problem as follows

$$y''(x) = y^3 - yy', \quad 1 \le x \le 2, \quad y(1) = \frac{1}{2}, \text{ and } y(2) = \frac{1}{3}, \tag{3.7}$$

with exact solution given by

$$y(x) = \frac{1}{(x+1)}.$$

a) Shooting with secant method

Same as Problem 1, we reduced (3.7) into the form of

$$w''(x) = w^3 - ww', \quad 1 \le x \le 2, \quad w(1) = \frac{1}{2}, \quad w'(1) = v_k, \tag{3.8}$$

and expressed (3.8) as first-order differential equation where

$$u = w', \quad u' = w^3 - wu, \quad w(1) = \frac{1}{2}, \quad u(1) = v_k. \tag{3.9}$$

Then, (3.9) is solved with two initial guess $v_0 = 0.25,$ and $v_1 = 0.45$ before proceed to find the new value for v_k. Problem 2 needs three iteration for $\phi(v_k)$ to converge within the given error tolerance.

Table 4: Approximate solution of Problem 2 with different values of v_k (Secant)

$w(i)$	$v_0 = 0.25$	$v_1 = 0.45$	$v_2 = -0.2532$	Exact Solution
1.0	0.5000	0.5000	0.5000	0.5000
1.1	0.5250	0.5445	0.4759	0.4762
1.2	0.5502	0.5883	0.4539	0.4545
1.3	0.5756	0.6315	0.4339	0.4348
1.4	0.6015	0.6746	0.4155	0.4167
1.5	0.6279	0.7178	0.3985	0.4000
1.6	0.6552	0.7616	0.3829	0.3846
1.7	0.6835	0.8064	0.3683	0.3704
1.8	0.7129	0.8528	0.3548	0.3571
1.9	0.7439	0.9014	0.3422	0.3448
2.0	0.7766	0.9528	0.3304	0.3333

b) Shooting by Newton Raphson's method

Same with Problem 1, we write the expression in (2.10) for Problem 2 where

$$z'' = (3w^2 - u)z + (-w)z', \quad 1 \le x \le 2, \quad z(1) = 0, \quad z'(1) = 1. \tag{3.10}$$

We then expressed (3.10) in the form of first-order differential equation where

$$t = z', \quad t' = (3w^2 - u)z + (-w)t, \quad z(1) = 0,\ t(1) = 1. \tag{3.11}$$

Solving (3.9) and (3.11) simultaneously with $v_0 = 0.25$ to obtain the value of $w(b, v_0)$, we then can proceed with the process of choosing new v_k using formula in (2.11). Through shooting with Newton's, Problem 2 needs two iteration for the $w(b, v_k)$ to converge within the provided tolerance error.

Table 5: Approximate solution of Problem 1 with different values of v_k (Newton's)

$w(i)$	$v_0 = 0.25$	$v_1 = -0.2541$	Exact Solution
1.0	0.5000	0.5000	0.5000
1.1	0.5250	0.4758	0.4762
1.2	0.5502	0.4538	0.4545
1.3	0.5756	0.4336	0.4348
1.4	0.6015	0.4151	0.4167
1.5	0.6279	0.3981	0.4000
1.6	0.6552	0.3824	0.3846
1.7	0.6835	0.3678	0.3704
1.8	0.7129	0.3542	0.3571
1.9	0.7439	0.3415	0.3448
2.0	0.7766	0.3296	0.3333

c) Solution of Problem 2 using FDM

Table below shows the solution of Problem 2 solved by FDM.

Table 6: Numerical solution of Problem 2 using FDM

$w(i)$	FDM Solution	Exact Solution
1.0	0.5000	0.5000

1.1	0.4762	0.4762
1.2	0.4546	0.4545
1.3	0.4348	0.4348
1.4	0.4167	0.4167
1.5	0.4000	0.4000
1.6	0.3846	0.3846
1.7	0.3704	0.3704
1.8	0.3571	0.3571
1.9	0.3448	0.3448
2.0	0.3333	0.3333

3.3 Discussion

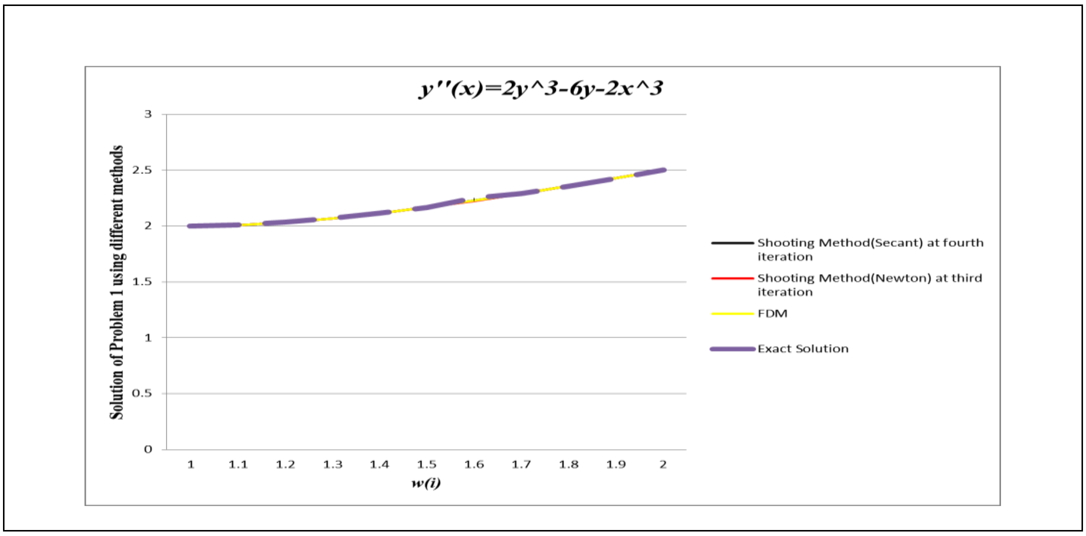

Figure 1: Solution of Problem 1 using different method

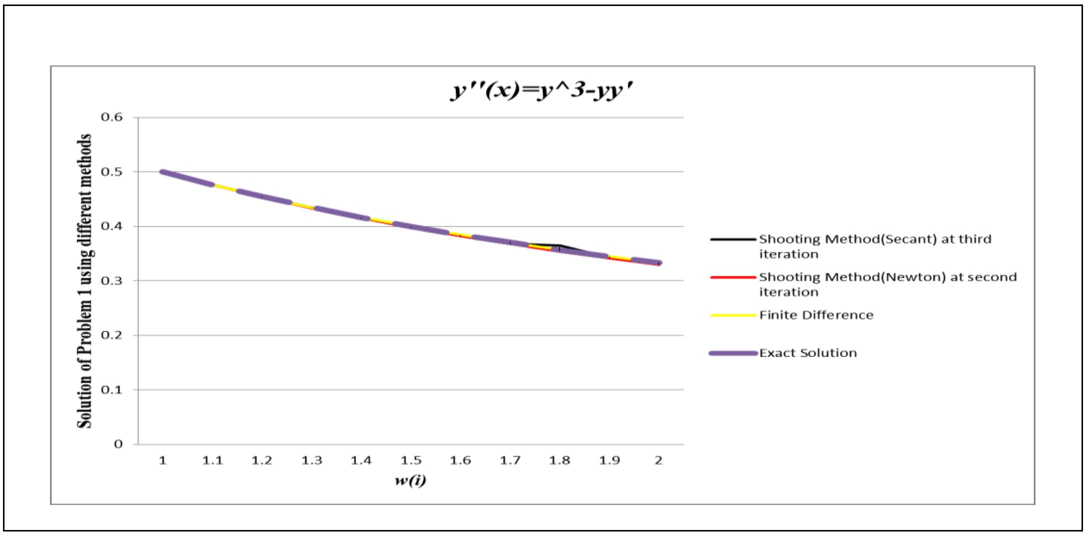

Figure 2: Solution of Problem 2 using different method

We can see from 3.1 and 3.2 that the result obtained through the implementation of shooting method based on secant and Newton Raphson's were slightly differ with FDM and exact solution. From both problem, we can observe that shooting based Newton Raphson's method required less iteration to get the same accuracy as shooting with secant and problems' exact solution. However, FDM outperforms both shooting method in term of iteration needs to get the same accuracy as exact solution which shows that FDM converges faster than shooting method in solving non-linear two-point boundary value problems.

4. Conclusion

From the results obtained in 3.1 and 3.2, shooting method proved to be efficient in solving a non-linear two-point boundary value problems. However, finite difference method converges using less iteration in obtaining the same accuracy as exact solution compared to shooting method. To increase the accuracy of shooting method, we can implement a smaller step size of h in order to obtain approximate solution close to the exact solution. In the future, we recommend to implement a modified shooting method in order to enhance the accuracy of approximate solution for the problems. An improvement on the algorithm of shooting method in using simple term to obtain result close to exact solution also can be apply in the further study for shooting method.

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