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Modified Decomposition Method for Solving A Nonlinear System of Two-dimensional Volterra-Fredholm Integral Equation

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Abstract: In this study, Modified Decomposition Method (MDM) is used to solve the nonlinear system of two-dimensional Volterra-Fredholm integral equations. The solution of the approximate equation is calculated with the help of Maple software. This method is proved to reduce computational works and simple to apply. The numerical results obtained in MDM are compared with the exact solution to see the accuracy of the method. A comparison of the absolute errors between MDM and Successive Iterative Method shows that MDM is more efficient than other methods.

Keywords: Nonlinear System of Two-dimensional Volterra-Fredholm Integral Equation, Modified Decomposition Method, Successive Iterative Method

1. Introduction

This paper presents the MDM for solving a nonlinear system of two-dimensional Volterra-Fredholm integral equations of second kind. The nonlinear mixed Volterra–Fredholm integral equation in two variables is given in [2] as

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega k(x,t,y,z,u(y,z))\,\mathrm{d}y\mathrm{d}z, \quad (x,t) \in [0,T] \times \Omega \tag{1.1}$$

where $f(x,t)$ is the source function and $k(x,t,y,z,u)$ is the kernel function that are the given analytical functions defined on $D := [0,T] \times \Omega$ and Ω is a closed subset of $(\Re^n, n = 1,2,3)$.

The topic of integral equations is one of the most important mathematical tools for both pure and applied mathematics. It has immense applications for many physical problems. The Volterra-Fredholm integral equations arise from problems of parabolic boundary value, mathematical modeling of the spatial-temporal evolution of the epidemic, and various physical and biological models. The essential

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features of these models are widely applicable. The Modified Decomposition Method (MDM) is the reliable modification of the Adomian Decomposition Method (ADM) was developed by Wazwaz [3]. The MDM can be applied to all integral equations and differential equations of any order and either linear or nonlinear equation.

The solution of nonlinear equation systems is perhaps one of the most difficult problems in all numerical calculations, especially in a wide range of engineering applications [5]. The convergence and performance characteristics can be highly sensitive to the initial assumption of the solution for most numerical methods. In previous research, numerical method such as Projection Methods, Time Collocation Method, Trapezoidal Nystrom Method, ADM and some else are used to solve this problem [3]. Then, when they apply these methods to the problem, it is becoming difficult to solve and sometimes it may consume a lot of time in getting approximate to exact solution. Therefore, in this study, we will explore analytical method in order to determine the effectiveness of the propose method. MDM is suitable method that we will use, as this method has approved by many mathematicians as fast convergence method compared to the other classical method and the series of solution using MDM will converge to the exact solution. The purpose of this research are to apply MDM on the nonlinear system of two-dimensional Volterra-Fredholm integral equations and to compare the absolute error between of MDM and Successive Iterative Method.

2. Methodology

Wazwaz [2] had first introduced the MDM. When applied to general nonlinear equation, the principal algorithm of the ADM is in the form

$$Lu+Ru+Nu=g(x), \tag{2.1}$$

where L is the highest order derivative that is considered to be conveniently invertible, R is lesser-order linear differential operator than L, Nu represents the nonlinear terms, and $g(x)$ is the source term.

The inverse operator L^{-1} is applied to both sides of (2.1), and using the given conditions we obtain,

$$u=f(x)-L^{-1}(Ru)-L^{-1}(Nu), \tag{2.2}$$

where the function $f(x)=L^{-1}(g(x))$.

The nonlinear operator $Nu=G(u)$ is decomposed as

$$G(u)=\sum_{n=0}^{\infty}A_n, \tag{2.3}$$

Adomian Polynomials were given in the algorithms,

$$\begin{aligned}
&A_0=G(u_0),\\
&A_1=u_1G'(u_0),\\
&A_2=u_2G'(u_0)+\frac{1}{2!}u_1G''(u_0),\\
&A_3=u_3G'(u_0)+u_1u_2G''(u_0)+\frac{1}{3!}u_1^{\,3}G'''(u_0),\\
&\ \vdots
\end{aligned} \tag{2.4}$$

2.1 Adomian Decomposition Method

The ADM involves of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of elements described in the decomposition sequence

$$u(x)=\sum_{n=0}^{\infty}u_n(x), \tag{2.5}$$

where the components $u_0(x), u_1(x), u_2(x), \ldots$are determined recursively and initial component $u_0(x)$ is the all terms not included under the integral sign. The components $u_j(x),\ j\geq 1$ of the unknown function $u(x)$ are then completely defined by setting the recurrence relation:

$$\begin{aligned} &u_0(x)=f(x),\\ &u_{n+1}(x)=-L^{-1}(Ru_n(x))-L^{-1}(A_n), \quad n\geq 0, \end{aligned} \tag{2.6}$$

Substitution (2.4) into (2.6) leads to the determination of the $u(x)$ component. Determining the components $u_0(x), u_1(x), u_2(x), \ldots$the solution $u(x)$ in a series form defined by (2.5) immediately follows [1].

2.2 Modified Decomposition Method

To apply this modification, Wazwaz [2] assumed that the function $f(x)$ can be divided into the sum of two part, defined by

$$f(x)=f_1(x)+f_2(x) \tag{2.7}$$

where $f_1(x)$ contain the zeroth component $u_0(x)$, while $f_2(x)$ is the remaining terms. Thus, the MDM introduces the modified recurrence relation:

$$\begin{aligned} &u_0(x)=f_1(x),\\ &u_1(x)=f_2(x)-L^{-1}(Ru_0(x))-L^{-1}(A_0),\\ &u_{n+1}(x)=-L^{-1}(Ru_n(x))-L^{-1}(A_n), \quad n\geq 1. \end{aligned} \tag{2.8}$$

Based on the description above, it shows that the difference recurrence relation between ADM and MDM. The zeroth component $u_0(x)$ of ADM in (2.6) is defined by the function $f(x)$, while zeroth component $u_0(x)$ for the MDM is defined by selecting some terms of function $f(x)$ and the remaining part will be added to the next component $u_1(x)$.

Note that, the exact solution $u(x)$ can be obtained by correctly choosing the functions $f_1(x)$ and $f_2(x)$ using very few iterations, and often by evaluating only two components. The efficiency of this modification depends only on the right selection of $f_1(x)$ and $f_2(x)$, and this can only be achieved by trials. A rule that may assist in the correct choice of $f_1(x)$ and $f_2(x)$ has not yet been discovered [2].

2.3 The MDM Applied to The Nonlinear System of Two-dimensional Volterra-Fredholm Integral Equation

Consider the nonlinear two-dimensional Volterra-Fredholm integral equation as follows

$$u(x,t)=f(x,t)+\int_0^t\int_{\Omega}k(x,t,y,z,u(y,z))\mathrm{d}y\mathrm{d}z\ , \quad (x,t)\in[0,T]\times\Omega, \tag{2.9}$$

where $f(x,t)$ is the source function and $k(x,t,y,z,u)$ is the kernel function that are the given analytical functions defined on $D:=[0,T]\times\Omega$ and Ω is a closed subset of $(\Re^n, n=1,2,3)$.

Substituting the decomposition series (2.5) into (2.9) yield

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x,t) + \int_0^t\int_\Omega k(x,t,y,z)\left(\sum_{n=0}^{\infty} u_n(y,z)\right)\mathrm{d}y\mathrm{d}z \ , \tag{2.10}$$

to apply this modification, the function $f(x,t)$ can be divided into the sum of two part, defined by

$$f(x,t) = f_1(x,t) + f_2(x,t) \tag{2.11}$$

where $f_1(x,t)$ contain the zeroth component $u_0(x,t)$,while $f_2(x,t)$ is the remaining terms. Thus, the MDM introduces the modified recurrence relation:

$$\begin{aligned} u_0(x,t) &= f_1(x,t), \\ u_1(x,t) &= f_2(x,t) + \int_0^t\int_\Omega k(x,t,y,z,u_0(y,z))\ \mathrm{d}y\mathrm{d}z, \\ u_{k+1}(x,t) &= \int_0^t\int_\Omega k(x,t,y,z,u_k(y,z))\ \mathrm{d}y\mathrm{d}z,\ k\geq 1. \end{aligned} \tag{2.12}$$

Relation (2.12) will enable us to determine the components $u_n(x,t),\quad n\geq 0$ recurrently, and as a result, the series solution of $u(x,t)$ is readily obtained.

3. Results and Discussion

This section will illustrate the Modified Decomposition Method (MDM) by solve several examples. The first example is a two-dimensional Fredholm integral equation, second is a two dimensional Volterra integral equation, and last example is a two dimensional Volterra-Fredholm integral equation. The examples of this equations and absolute error of Successive Iterative Method are referred to the article by Borzabadi and Heidari [4]. Next, we make the comparison the absolute error of MDM and Successive Iterative Method to exact solution. The result obtained demonstrate the efficiency and accuracy of the MDM to exact solution.

3.1 Example 1

A two-dimensional Fredholm integral equation as follow [4]:

$$u(x,t) = \frac{1}{(1+x+t)^2} - \frac{x}{6(1+t)} + \int_0^1\int_0^1 \frac{x}{1+t}(1+y+z)u^2(y,z)\ \mathrm{d}y\mathrm{d}z. \tag{3.1}$$

Analytical solution of [4, eq. (3.1)] is

$$u(x,t) = \frac{1}{(1+x+t)^2}\ \text{ on } [0,1]\times[0,1]. \tag{3.2}$$

The numerical solutions are obtained and the 3d-graph of the solution is plotted by using maple software.

Table 3.1: Estimated and exact value of $u(x,t)$ for example 1.

$t = x$	MDM	Exact solution
0.0	1.000	1.000
0.2	5.102×10^{-1}	5.102×10^{-1}
0.4	3.086×10^{-1}	3.086×10^{-1}
0.6	2.066×10^{-1}	2.066×10^{-1}
0.8	1.479×10^{-1}	1.479×10^{-1}
1.0	1.111×10^{-1}	1.111×10^{-1}

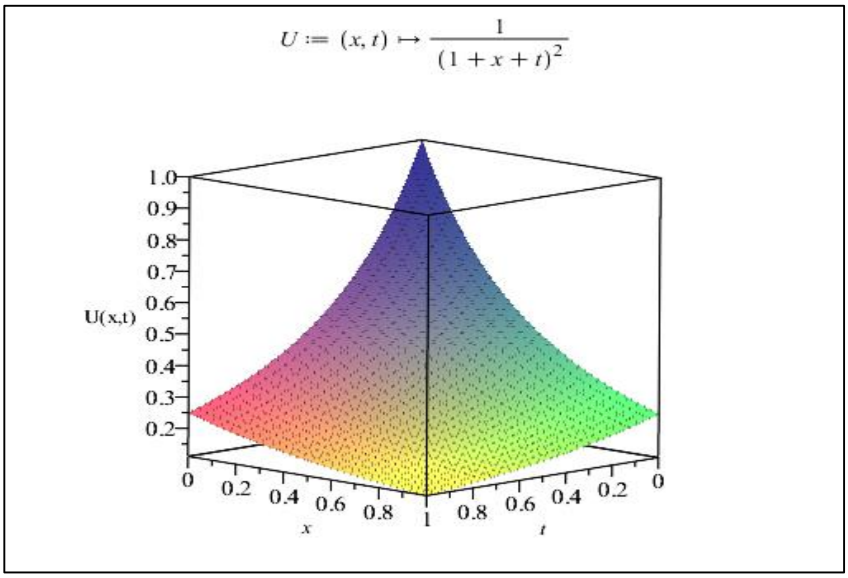

Figure 3.1: 3D graph of numerical solution for example 1

Table 3.2: Absolute error of methods to exact solution for example 1

$t = x$	MDM	Successive Iterative Method
0.0	0.000	0.000
0.2	0.000	3.317×10^{-6}
0.4	0.000	5.684×10^{-6}
0.6	0.000	7.457×10^{-6}
0.8	0.000	8.835×10^{-6}
1.0	0.000	9.937×10^{-6}

Based on the Table 3.2, the absolute error equal to zero for MDM while Successive Iterative Method are not. This shows that the MDM is more efficiency and accuracy to exact solution for this example.

3.2 Example 2

A two-dimensional Volterra integral equation as follow [4]:

$$u(x,t) = x\sin(t)\left(1-\frac{x^2\sin^2(t)}{9}\right)+\frac{x^2}{10}\left(\frac{\sin(2t)}{2}-t\right)+\int_0^t\int_0^x (xy^2+\cos(z))u^2(y,z)\,\mathrm{d}y\mathrm{d}z. \quad (3.3)$$

Analytical solutions of [4, eq. (3.3)] is

$$u(x,t) = x\sin(t) \text{ on } [0,1]\times[0,1]. \quad (3.4)$$

The numerical solutions are obtained and the 3d-graph of the solution is plotted by using maple software.

Table 3.3: Estimated and exact value of $u(x,t)$ for example 2.

t=x	MDM	Exact solution
0.0	0.000	0.000
0.2	3.973×10^{-2}	3.973×10^{-2}
0.4	1.557×10^{-1}	1.557×10^{-1}
0.6	3.387×10^{-1}	3.387×10^{-1}
0.8	5.738×10^{-1}	5.738×10^{-1}
1.0	8.414×10^{-1}	8.414×10^{-1}

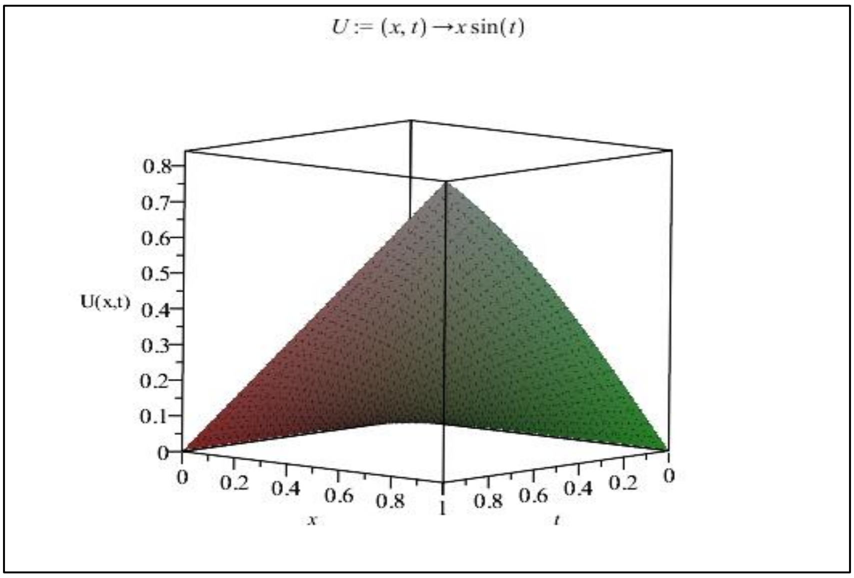

Figure 3.2: 3D graph of numerical solution of example 2

Table 3.4 Absolute error of methods to exact solution for example 2

x=t	MDM	Successive Iterative Method
0.0	0.000	3.906×10^{-31}
0.2	0.000	1.722×10^{-8}
0.4	0.000	2.616×10^{-7}
0.6	0.000	1.338×10^{-6}
0.8	0.000	4.852×10^{-6}
1.0	0.000	1.571×10^{-5}

Based on the Table 3.4, the absolute error equal to zero for MDM while Successive Iterative Method are not. This shows that the MDM is more efficiency and accuracy to exact solution for this example.

3.3 Example 3

A two-dimensional Volterra-Fredholm integral equation as follow [4]:

$$u(x,t) = x^2 + xt - \frac{1}{15}xt^4 - \frac{1}{16}xt^5 + \int_0^t\int_0^1 (xty^2z^2)(u(y,z))\,\mathrm{d}y\mathrm{d}z. \tag{3.5}$$

Analytical solution of [4, eq. (3.5)] is

$$u(x,t) = x^2 + xt \text{ on } [0,1]\times[0,1]. \quad (3.6)$$

The numerical solutions are obtained and the 3d-graph of the solution is plotted by using maple software.

Table 3.5: Estimated and exact value of $u(x,t)$ for example 1

x=t	MDM	Exact solution
0.0	0.000	0.000
0.2	0.080	0.080
0.4	0.320	0.320
0.6	0.720	0.720
0.8	1.280	1.280
1.0	2.000	2.000

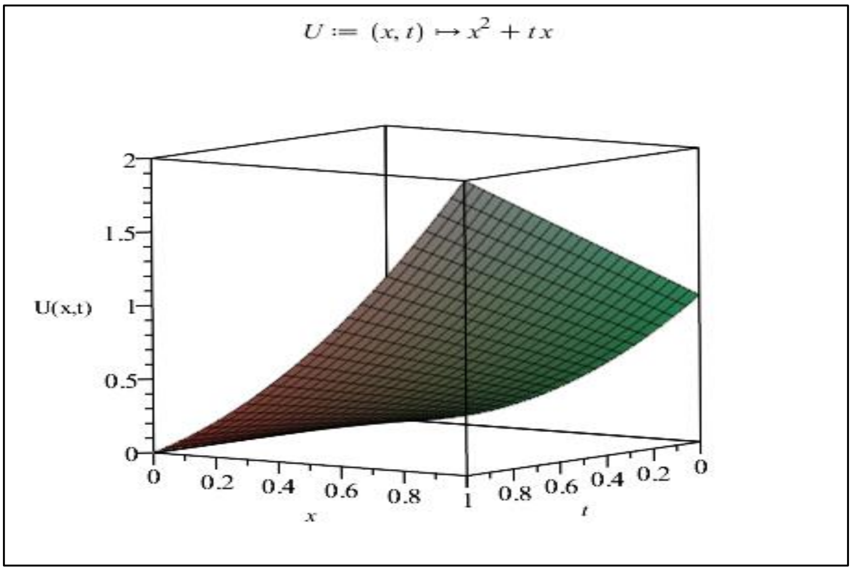

Figure 3.3: 3D graph of numerical solution of example 3

Table 3.6 Absolute error of methods to exact solution for example 3

x=t	MDM	Successive Iterative Method
0.0	0.000	0.000
0.2	0.000	4.063×10^{-8}
0.4	0.000	5.133×10^{-7}
0.6	0.000	2.700×10^{-6}
0.8	0.000	9.701×10^{-6}
1.0	0.000	2.799×10^{-5}

Based on the Table 3.6, the absolute error equal to zero for MDM while Successive Iterative Method are not. This shows that the MDM is more efficiency and accuracy to exact solution for this example.

3.4 Discussion

Based on the results obtained by MDM, we get the same solution as the exact solution. Next, this method needs a few iterations to obtain the approximate solution and this proves that this modification has been accelerating the convergence of the solution and minimizing the size of computational work. The success of this modification depends on the proper choice of $f_1(x,t)$ and $f_2(x,t)$, and this can only be achieved by trials. The comparison of absolute error between MDM and Successive Iterative Method shows that MDM is more efficient and accurate to the exact solution for the examples in previous section. Maple 15 is the software has been used for the calculation and plotted the 3D graph. It is convenient for solving the integration in Volterra-Fredholm integral equation.

4. Conclusion

This paper is the MDM for solving a nonlinear system of two-dimensional Volterra-Fredholm integral equation. Based on the results obtained, we can see that MDM is identical with exact solution for the given problems considered and fast convergence. In conclusion, the MDM is the efficient way for solving the nonlinear of two-dimensional Volterra-Fredholm integral equation and more accuracy to exact solution than Successive Iterative Method. Thus, the objectives of this research have been achieved. The recommendations that could be made for further review and research on MDM are make the right function $f(x)$ separation to get a solution quickly and accurately and this can be done through experimentation. Besides, the standard decomposition method can be used if the function $f(x)$ consists of only one term.

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