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Solving Lighthill-Whitham and Richards Traffic Flow Model using Discontinuous Galerkin Method

Kueh Hui Ping[1], Siti Suhana Jamaian[1*]

[1] *Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh Hab Pendidikan Tinggi Pagoh KM 1, Jalan Panchor, 84600 Pagoh, Muar, Johor, MALAYSIA.*

*Corresponding Author: suhana@uthm.edu.my
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Abstract

Traffic congestion at junctions, particularly in urban areas, presents significant challenges in Malaysia. Addressing this issue, the study provides an efficient numerical approach for solving the Lighthill-Whitham and Richards (LWR) traffic flow model using the discontinuous Galerkin (DG) method. The scope of this study is limited to the LWR model at junctions with one incoming and two outgoing roads. The study aims to derive the numerical flux of the LWR model, especially focusing on junction scenario using the Lax-Friedrichs numerical flux and hence solve the LWR model numerically using the DG method. The results demonstrate the DG method's ability to accurately capture sharp discontinuities and provide a reliable representation of traffic behaviours at junctions.

1. Introduction

Traffic congestion at urban junctions poses significant challenges in Malaysia, exacerbated by rising vehicular use. A junction is where multiple roads meet and has a critical point of traffic congestion especially when there is no traffic control measures such as traffic lights. Drivers entering or leaving the junctions face delays as they have to wait until there are no vehicles from the outgoing roads and it will be worse during a peak hour which will lead to traffic bottlenecks and variability in flow [1].

Previous research often focuses on general traffic network modelling without addressing the specific complexities of junctions, for example [2-5]. Therefore, the study addresses such issues through the first-order macroscopic traffic flow model, Lighthill-Whitham and Richards (LWR) model, solved numerically using the discontinuous Galerkin (DG) method.

The traffic flow models are widely used to analyse and predict traffic behaviour and offer valuable insights for managing congestion. These models can be categorized into three main types: microscopic, mesoscopic and macroscopic. According to [6], microscopic models simulate the behaviour of individual vehicles and their interactions while [7] stated that mesoscopic models incorporate characteristics of both microscopic and macroscopic approaches. As in [8], macroscopic traffic flow models, such as the first-order macroscopic model, the LWR model see the traffic as a continuum and focus on the relationship between the traffic flow, traffic density and speed.

The LWR model was introduced independently by Lighthill and Whitham in 1955 and Richards in 1956 [9]. It is a first-order macroscopic traffic flow model based on scalar conservation laws. It is more effective for modelling traffic flow at a large scale as it captures special situations such as shock waves and congestion propagation. The LWR traffic flow model provides a framework to analyse and predict congestion. These models describe traffic as a continuum and are valuable for understanding macroscopic traffic dynamics including traffic density and traffic flow according to [10]. In addition, in [2] the LWR model describes the relationship between the traffic density and the traffic flow. It is particularly suitable for analysing traffic at a macroscopic level, capturing phenomena like shock waves and congestion propagation.

Numerical methods are critical for solving the traffic flow models especially when the analytical solutions are impractical. However, many numerical methods have difficulty capturing the sharp discontinuities in traffic density and traffic flow that occur at these points. This gap highlights the need for a focused approach to accurately model traffic behaviour at junctions, ensuring more reliable insights into congestion and traffic flow dynamics.

The DG method was developed in the 70s. It has gained prominence for its ability to handle problems with sharp gradients and discontinuities. It combines the features of finite volume and finite element methods in [11] and makes it suitable for traffic situations that have abrupt changes like congestion [12]. Recent studies have demonstrated the DG method's effectiveness in solving the LWR model for road networks when paired with numerical flux such as Lax-Friedrichs flux in [13].

A lot of researchers have applied the LWR model to general traffic networks, however, the studies that focused on junctions are limited. Junctions present unique challenges such as varying traffic densities and flow distributions between multiple roads. Hence, the paper specifically explores traffic scenarios at junctions with one incoming and two outgoing roads. According to [11], by using the Lux-Friedrichs numerical flux, the study derives the LWR model's numerical flux that involves the incoming road and outgoing road at the junction scenario. Besides, the study includes three examples by finding the difference between the traffic density and traffic flow during a normal period, with a heavy traffic density and lastly during a peak hour.

The studies by [11], [14], [15] have addressed similar issues using higher-order macroscopic models such as the Payne-Whitham models or Aw-Rascle models, but these approaches often involve greater complexity and computational costs. Therefore, this study is aimed to apply the DG method to the LWR model at the junction with one incoming road and two outgoing roads which will highlight the DG method's potential for accurate and efficient traffic modelling.

2. Methodology

This study applies the DG method to solve the LWR traffic flow model at a junction with one incoming road and two outgoing roads. From [13], the LWR traffic flow model can be defined as (1).

$$\rho_t + f(\rho)_x = 0 \tag{1}$$

where ρ_t represents the partial derivative of $\rho(x,t)$ with respect to t and $f(\rho)_x = \frac{\partial[\rho(x,t)\, v(\rho(x,t))]}{\partial x}$, while $v(\rho(x,t))$ is a velocity function giving the fundamental relation between the density and the velocity of vehicles. The Greenshields model defines the flow function as (2).

$$v_e(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) \tag{2}$$

where $v_e(\rho)$ is the equilibrium velocity. This formulation describes the linear relationship between traffic density and equilibrium velocity.

By using the DG method to solve the LWR model, first multiply the LWR model by a test function ϕ and obtain (3).

$$\phi\rho_t + \phi f(\rho)_x = 0 \tag{3}$$

Then, integrate (3) for an arbitrary element $K \in \tau_h$ by using integration by parts and get (4).

$$\int_\rho \rho_t \phi dx - \int_\rho f(\rho)_x \phi' dx + f\left(\rho\left(b_{p,t}\right)\right)\phi^{(L)} - f\left(\rho\left(a_{p,t}\right)\right)\phi^{(R)} = 0 \tag{4}$$

Finally, summarize all $\rho \in \tau_h$ to obtain (5).

$$\int_\Omega \rho_t \phi dx - \sum_{\rho \in \tau_h} \int_\rho f(\rho)_x \phi' dx + \sum_{x \in \mathscr{F}_h} f(\rho)[\phi] = 0 \tag{5}$$

To approximate ρ by the function $\rho_h \in H^1\left(\Omega, \tau_h\right)$ which is in general discontinuous on $\mathscr{F}_h$, the approximation of the function $f\left(\rho_h\right)_x$ in the points $x \in \mathscr{F}_h$ can be written as (6).

$$f\left(\rho_h\right)_x \approx H\left(\rho_h^{(L)}, \rho_h^{(R)}\right) \tag{6}$$

where $H\left(\rho_H^{(L)}, \rho_h^{(R)}\right)$ is a numerical flux and (L) represents the left-hand side of the outgoing road while (R) represents the right-hand side of the incoming road.

Definition 1 (DG solution)
According to [11], the function $u_h: \Omega \times (0,T) \to \mathbb{R}$ is called a discontinuous Galerkin finite element solution of a hyperbolic problem as the following (7), (8) and (9) if the properties hold.

$$\rho_t + f(\rho)_x = 0, \quad x \in \Omega, \quad t \in (0,T) \tag{7}$$

$$\rho = \rho_D, \quad x \in \mathscr{F}_h^D \times t \in (0,T) \tag{8}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \Omega \tag{9}$$

where the Dirichlet boundary condition $\rho_D: \mathscr{F}_h^D \times (0,T) \to \mathbb{R}$ and the initial condition $\rho_0: \Omega \to \mathbb{R}$ are the given functions. The properties that should be held is as follows.

i. $\rho_h \in C^1\left(\left[0,T\right]; S_h\right)$.
ii. $\rho_h(0) = \rho_{h0}$ where ρ_{h0} is an S_h approximation of the initial condition ρ_0.
iii. $\rho_h = \rho$ for all $x \in \mathscr{F}_h^D$, $t \in (0,T)$.

For all $\phi \in S_h$ and for all $t \in (0,T)$, ρ_h satisfies (10).

$$\int_\Omega \rho_t \phi dx - \sum_{\rho \in \tau_h} \int_K f(\rho)_x \phi' dx + \sum_{x \in \mathscr{F}_h} H\left(\rho_h^{(L)}, \rho_h^{(R)}\right)[\phi] = 0 \tag{10}$$

To model the traffic flow models at junctions, the numerical flux at boundaries is defined using the Lax-Friedrichs flux is shown in (11).

$$H\left(\rho_h^{(L)}, \rho_h^{(R)}\right) = \frac{1}{2}\left[f\left(\rho_h^{(L)}\right) + f\left(\rho_h^{(R)}\right) - \alpha\left(\rho_h^{(R)} - \rho_h^{(L)}\right)\right] \tag{11}$$

where $f(\rho)$ represents the flow function and α is the maximum wave speed. Consider that I_i as the incoming road and I_j is the outgoing road where they are directly connected, then the numerical flux (11) can be simplified to (12).

$$H\left(\rho_{hi}^{(L)}\left(b_i, t\right), \rho_{hj}^{(R)}\left(a_j, t\right)\right) \tag{12}$$

where ρ_{hi} and ρ_{hj} are the DG solutions on I_i and I_j respectively. After that, separate (12) into the left-hand side of the outgoing road and right-hand side for the incoming road. Let $H_i(t)$ be the numerical flux for right-hand side of the incoming road at the junction and $H_j(t)$ be the numerical flux for the left-hand side of the outgoing road, (13) represents the solution for $H_i(t)$ while (14) represents the solution for $H_j(t)$.

$$H_i(t) := \sum_{i=1+n}^{n+m} \alpha_{j,i} H\left(\rho_i^{(L)}\left(b_i, t\right), \rho_j^{(R)}\left(a_j, t\right)\right), \tag{13}$$

for $i = 1, 2, \ldots, n$ as n represents the number of outgoing roads.

$$H_j(t) := \sum_{i=1}^{n} \alpha_{j,i} H\left(\rho_i^{(L)}\left(b_i, t\right), \rho_j^{(R)}\left(a_j, t\right)\right), \tag{14}$$

for $j = n+1, n+2, \ldots, n+m$ as m represents the number of outgoing roads.

3. Results and Discussions

In this study, the derivation of the numerical flux for the LWR model at the junction was successfully derived by using the Lax-Friedrichs flux in Section 2. Besides that, the LWR model has been solved numerically by using the DG method and conducted using Python software to produce illustrations that provide valuable insights for this study.

This study considers three different situations for the LWR traffic flow model such as during a normal period, with heavy traffic density and lastly during peak hour to make a comparison of the illustrations of the traffic density and traffic flow. According to [3], the Greenshields model was used in this study as a guideline for the maximum density and maximum velocity where $v_{max} = \rho_{max} = 1$.

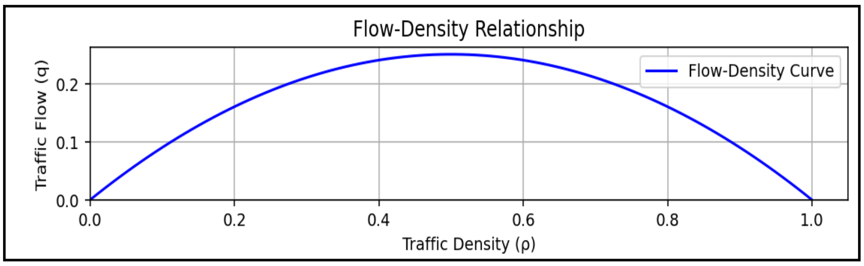

Fig. 1 *Traffic flow-density relationship*

Traffic flow is related to the number of vehicles passing a specific point of road per unit of time according to [3]. From Fig. 1, the traffic flow will increase when the traffic density is increased. However, when the traffic flow has reached its limit, it will decrease while the traffic density increases. For example, in a traffic jam or congestion, the traffic flow is the lowest as the vehicles move slowly or stop on the road and hence, there will be no vehicles passing through a point in a short time making the traffic density the highest.

This study considers the LWR traffic flow model at the junction with an incoming road and two outgoing roads. An incoming is where the vehicles come in a direction to arrive at a junction and is set as 10 km for this study. The outgoing roads at the junction are the roads where the vehicles go after turning left or right and outgoing road 1 is set as 10 km and outgoing road 2 is set as 15 km which helps to figure out the difference in traffic density and traffic flow for both roads.

The study discusses the LWR model in three situations: during a normal period, during heavy traffic density and peak hour. The normal period represents light traffic density where the number of vehicles is smaller than that in heavy traffic density and there is not any congestion on the roads. During heavy traffic density, the number of vehicles is smaller than the number of vehicles during peak hour but larger than that during normal period. However, during heavy traffic density, there will not be any congestion. At peak hour, the number of vehicles is huge and may reach the limit of the road which causes traffic jams or congestion. The illustrations of these three situations will be discussed.

3.1 Solution for the LWR model during a normal period

During a normal period, assume the number of vehicles $n = 20$ and the initial traffic density for the incoming road and outgoing road to be $\rho_i^{(L)} = 0.8$ veh km^{-1} and $\rho_j^{(R)} = 0.2$ veh km^{-1} respectively.

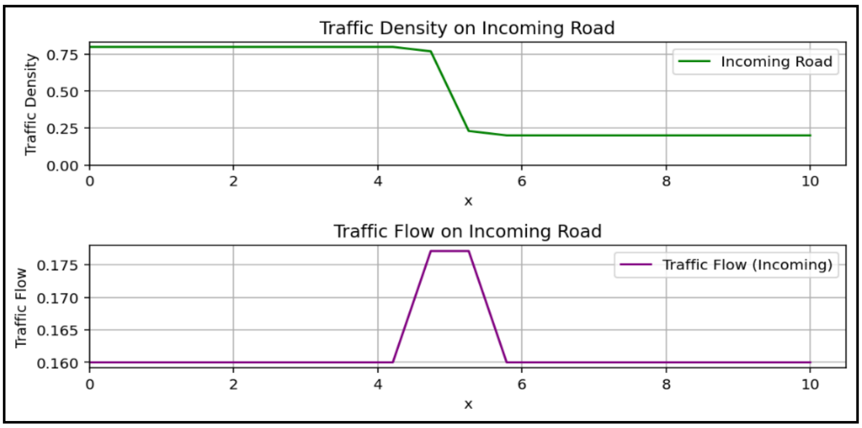

Fig. 2 Traffic density and traffic flow on incoming road during normal period

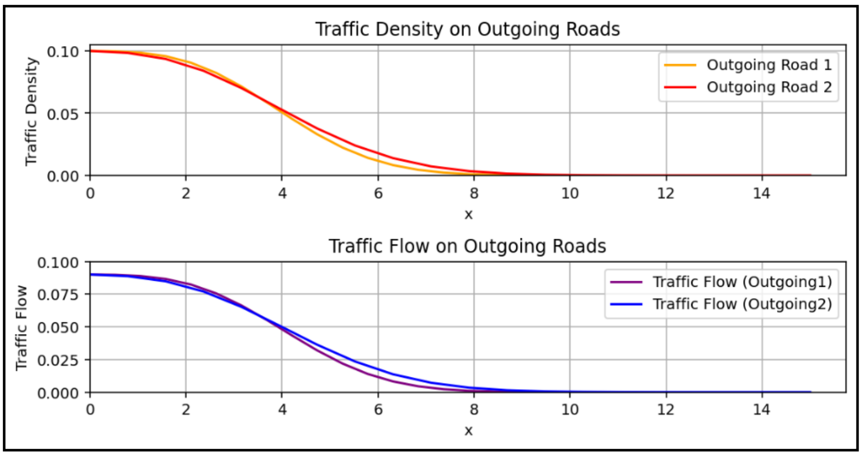

Fig. 3 Traffic density and traffic flow on the outgoing road 1 and 2 during normal period

From Fig. 2, the traffic density is 0.8 veh km^{-1} initially then rapidly decreases to 0.2 veh km^{-1}. This shows that the number of vehicles decreases and remains steady until the end of the incoming road. The decrease in traffic density may be caused by the decrease in vehicles entering the outgoing roads. For the traffic flow in Fig. 2, it is at a stable state at the beginning then reaches a peak above 0.175 veh h^{-1} then decreases again to its initial traffic flow when all the 20 vehicles go out and enter the outgoing roads. This shows the relationship between traffic flow and traffic density as the densities decrease, the traffic flow will be increased and hence explains the illustration in Fig. 1.

Fig. 3 shows the traffic flow and traffic density on both outgoing road 1 and 2 have the same trend where they generally decreased from a higher value to 0. The higher value of traffic density and traffic flow may result from the vehicles entering on the outgoing roads. Assuming there is no traffic light or bump on the junction, the values are decreased to 0 as no more vehicles are entering on outgoing road 1 and outgoing road 2.

3.2 Solution for the LWR model with heavy traffic density

The heavy traffic density represents the number of vehicles that increase to a certain number but do not have any congestion or traffic jams on the road. Hence, in this section, assume that the number of vehicles $n = 60$ while the initial traffic densities on the incoming road and traffic density on the outgoing roads are the same as in section 3.1.

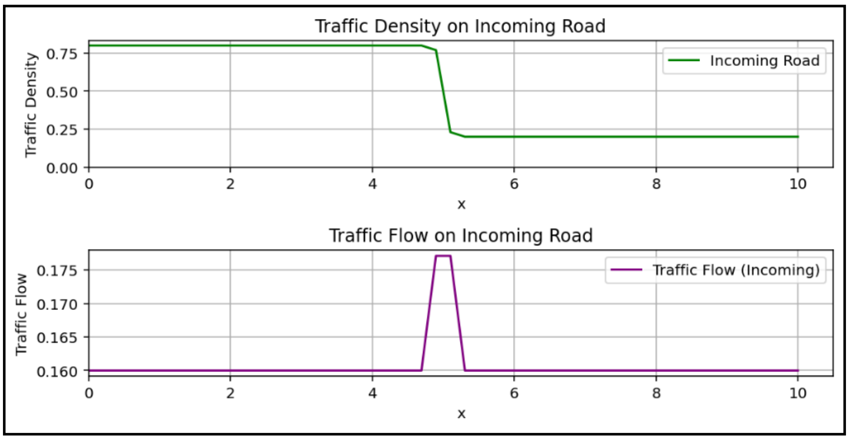

Fig. 4 Traffic density and traffic flow on incoming road with heavy traffic density

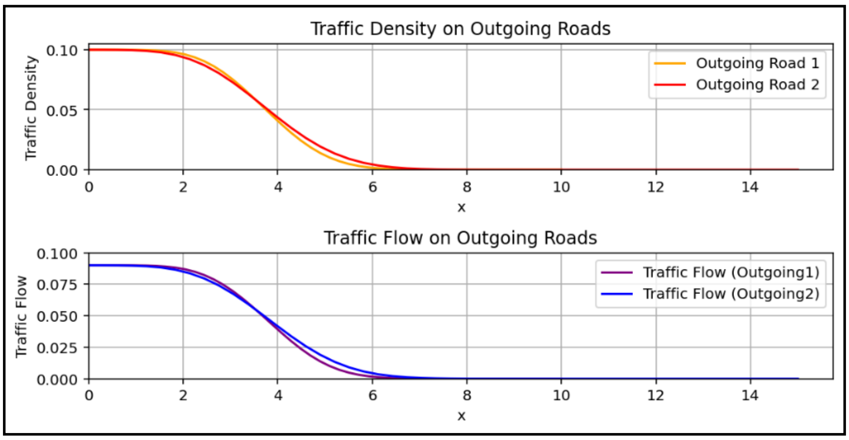

Fig. 5 Traffic density and traffic flow on outgoing roads with heavy traffic density

Fig. 4 and Fig. 5 show that they have the same illustration as Fig. 2 and Fig. 3. Fig. 4 shows that the traffic density on the incoming road has dropped from 0.8 veh km^{-1} to 0.2 veh km^{-1} at around $x = 5$ km. At the same time, the traffic flow in Fig. 4 has a sharper increase from 0.160 veh h^{-1} to above 0.175 veh h^{-1} compared to Fig. 2. This indicates that during a heavy traffic density situation, the vehicles will enter outgoing roads and cause the traffic density to decrease and simultaneously the traffic flow to be increased.

From Fig. 5, the traffic density and traffic flow on outgoing roads have a sharper decrease compared to Fig. 3. The traffic flow and traffic density in Fig. 5 decreased to 0 at $x = 6$ km while the traffic density and traffic flow in Fig. 3 decreased to the same value at $x = 8$ km. In summary, the LWR model with a heavy traffic density will cause the traffic density and traffic flow to be sharpened compared to those during the normal period.

3.3 Solution for the LWR model at peak hour

In this section, the same situation but at peak hour for example during rush hours or off hours is discussed. Assuming the number of vehicles entering the incoming road to be $n = 100$, the initial traffic density on the incoming road $\rho_i^{(L)} = 1$ veh km^{-1} and traffic density on the outgoing roads is $\rho_j^{(R)} = 0.2$ veh km^{-1}.

Fig. 6 Traffic density and traffic flow on incoming road at peak hour

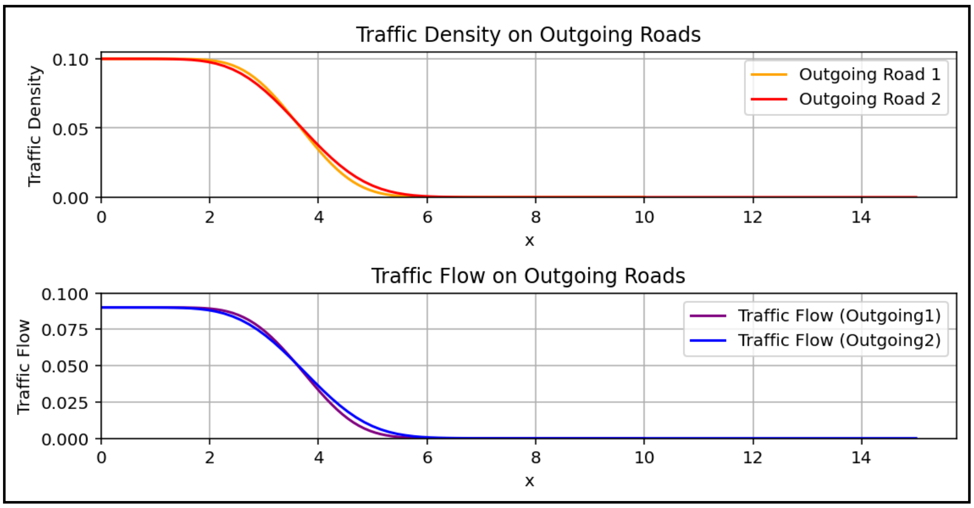

Fig. 7 Traffic density and traffic flow on outgoing roads at peak hour

From Fig. 6, noticed that the traffic density on the incoming road at peak hour has dropped from 1 veh km^{-1} to 0.2 veh km^{-1} at $x = 5$ km. However, the traffic flow has a sharp rise at around $x = 5$ km and then decreases to about 0.15 veh h^{-1} and lastly remains constant. This is because of the congestion or traffic jams as the vehicles, at the beginning, move slowly or stop and then accelerate to the outgoing roads. This explains that when the traffic density approaches the maximum number of vehicles on the road, the traffic flow is the lowest, then decreases when the traffic flow increases.

Fig. 7 shows that the traffic density and traffic flow for both outgoing road 1 and outgoing road 2 at peak hour are the same as they were in Fig. 3 and Fig. 5. This is because of the splitting of the vehicles from the incoming road to the outgoing roads.

In summary, the traffic density on the incoming road will reach a maximum at peak hour when the vehicles have to slow down or stop causing the traffic flow to be the lowest. When the vehicles disperse to different outgoing road 1 and outgoing road 2, the traffic density decreases and the traffic flow increases to a peak.

4. Conclusions and Recommendations

In conclusion, the objectives of this study have been achieved successfully. First, the derivation of the LWR model's numerical flux has been done by using the Lax-Friedrichs flux. It derived the numerical flux into two, each showing the numerical flux for the incoming road and outgoing road respectively.

Next, the numerical solution of the LWR model was computed using the DG method and Python software. The relationship between traffic density and traffic flow is shown in Fig. 1. During a normal period, the traffic density on the incoming road decreases while the traffic flow increases to a peak and then decreases to the initial value. For the traffic flow with a heavy traffic density, the traffic flow and traffic density have the same pattern as that during a normal period. However, they have a sharper decrease in traffic density and traffic flow compared to Fig. 3. From the opposite perspective, during a peak hour, the traffic density on the incoming road decreases with the same pattern as in Fig. 2 and Fig. 4. However, the traffic flow increases to a peak and does not decrease to the initial value before it remains constant at the end of the road because of the congestion on the traffic.

The results show that the DG method has effectively captured the sharp discontinuities which is the congestion at the junction while providing a high accuracy in the numerical solution. The results demonstrated a clear relationship between traffic density and traffic flow under varying conditions, including during normal period, heavy traffic and at peak hour.

However, there are still a lot of weaknesses that need to be overcome in future studies. The first one is this study does not indicate the traffic control at the junction. Future studies should indicate that traffic control such as traffic lights at the junction would be a better idea compared to assuming the number of vehicles on the outgoing roads to be zero.

In addition, future studies can focus on roundabout junctions as they would consist of more incoming and outgoing road junctions compared to one incoming road and two outgoing roads in this study. Future studies should consider different situations such as roundabouts with and without traffic lights. This would help to make a comparison between the traffic density and traffic flow for the roundabouts with and without traffic lights.

Lastly, future studies should use a higher-order macroscopic model for example the Payne-Whitham (PW) model as it can observe the nonequilibrium states of the traffic which the LWR model cannot. The PW model would give a better explanation of the nonequilibrium traffic although the solutions of the PW model at a stable are close to the solution of the LWR model.

In summary, there are some weaknesses in this study that should be overcome in future studies with the recommendations.

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Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Kueh Hui Ping, Siti Suhana Jamaian;* ***analysis and interpretation of results:*** *Kueh Hui Ping;* ***validation of results:*** *Siti Suhana Jamaian;* ***draft manuscript preparation:*** *Kueh Hui Ping, Siti Suhana Jamaian. All authors reviewed the results and approved the final version of the manuscript.*

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