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Differential Transform Method for Solving Ordinary Differential Equations with Variable Coefficients

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Abstract

Ordinary differential equations (ODEs) play a crucial role in various scientific and engineering fields, particularly in modelling physical, biological, and economic systems. The differential transform method (DTM) and the Sadiq-Emad-Jinan (SEJI) integral transform are two prominent techniques used to solve ODEs, each offering distinct advantages and limitations. This study aims to compare the accuracy and efficiency of these two methods when applied to solving ODEs with variable coefficients. The research applies DTM to selected second-order ODEs and analyses the solutions using numerical and graphical methods. Maple software is used to compute the transformed equations and obtain approximate and exact solutions. The study evaluates the precision, computational complexity, and applicability of both methods. Numerical analysis from test cases shows that while SEJI, being an analytical method, yields exact solutions, DTM provides highly accurate approximations but exhibits increasing absolute errors as the interval grows. Despite this, DTM remains computationally efficient and is more adaptable for nonlinear and large-scale problems compared to SEJI. The findings suggest that the choice between DTM and SEJI depends on the specific problem at hand. SEJI is more suitable for problems requiring exact solutions, while DTM is preferable for practical applications where computational efficiency and flexibility are priorities. By understanding the strengths and limitations of these methods, this study contributes valuable insights for researchers and engineers in selecting the most suitable approach for solving differential equations.

1. Introduction

Ordinary differential equations (ODEs) are also significant in the theory of oscillations and automatic control theory, where advancements in ODE theory have led to developments in optimal control [1]. The differential transform method (DTM), introduced by Zhou in 1986, is a semi-analytical approach designed for solving initial value problems in electrical circuits [2]. This method calculates the coefficients of the Taylor series expansion for the solution and has been extended to solve boundary value problems, integral equations, calculus of variations, and optimal control problems [2]. Unlike traditional Taylor series methods, which require computationally intensive derivative calculations, DTM provides an iterative procedure for obtaining higher-order solutions, making it well-suited for high-order equations [3].

DTM offers several advantages, including the ability to provide a closed-form solution as a series, high accuracy within its domain of convergence, low computational complexity for linear systems, and the absence of discretization errors [2]. However, it has limitations, such as requiring large memory for high-order computations, convergence restrictions for small domains, and challenges in handling nonlinear systems [4][5].

As in for Sadiq-Emad-Jinan (SEJI) integral transform, proposed in 2022, is a newly developed analytical method named after its creators' first names: Sadiq A. Mehdi, Emad A. Kuffi, and Jinan A. Jasim [6]. Closely related to the Laplace and Jafari transforms, it is defined in the time domain $t \geq 0$ for functions of exponential order. SEJI satisfies properties such as linearity, scaling, shifting, and the convolution theorem [6][7]. Although SEJI is still relatively new, studies have demonstrated its effectiveness in solving linear ODEs, partial differential equations (PDEs), delay differential equations, population growth models, and Bessel functions [6][8]. However, its application to nonlinear equations remains unexplored, and its general recognition in the mathematical community is limited due to its novelty.

DTM, despite its widespread use, has limitations, such as high memory consumption, intermediate complexity in Taylor series expansion, and convergence restrictions to small domains [2][4]. On the other hand, SEJI, as a new method, lacks extensive research and application, particularly for nonlinear problems [6]. Additionally, while numerous studies have compared DTM with other methods, no research has yet compared SEJI with established methods, leaving a gap in the literature [6][7].

2. Methodology

2.1 Differential Transform Method

The one-dimensional differential transform function is defined as in equation (1) [9]:

$$C(k) = \frac{1}{k!}\left[\frac{\delta^k}{\delta x^k} c(x)\right]_{x=0} \tag{1}$$

where $c(x)$ is the original function and $C(k)$ is the transformed function. The differential inverse transform of $C(k)$ is defined as in equation (2) [10]:

$$c(x) = \sum_{k=0}^{\infty} C(k) x^k \tag{2}$$

The fundamental operation performed by differential transformation as seen in Table 1 [11].

Table 1 Fundamental operations by DTM

Original function	Transformed function
$u(x) \pm v(x)$	$U(k) \pm V(k)$
$\alpha g(x)$	$\alpha G(k)$, where α is a constant
$\frac{d}{dt} g(x)$	$(k+1)U(k+1)$
$\frac{d^m}{dx^m} g(x)$	$(k+1)(k+2)\dots(k+m)G(k+m)$
x^m	$\delta(k-m)\begin{cases}1, k = m\\ 0, k \neq m\end{cases}$
$g(x)h(x)$	$\sum_{m=0}^{k} H(m)G(k-m)$
$e^{\lambda x}$	$\frac{\lambda^k}{k!}$

2.2 SEJI Integral Transform

Here present the general form and the transformed functions of SEJI integral transform according to [7]. Given an integrable function $f(t)$ under $t \geq 0, p(s) \neq 0$ and $q(s)$ are positive real functions, and i is a complex number. Assuming an integral exists for some $q(s)$, the general complex integral transformation $T_g^c(s)$ of $f(t)$ is defined as follows:

$$T_g^c\{f(t); s\} = F_g^c(s) = p(s) \int_{t=0}^{t=\infty} e^{iq(s)t} f(t) dt$$

Some important functions are transformed as in Table 2.

Table 2 Fundamental transformed functions by SEJI integral transform

Integrable function	Transformed function	Condition
1	$-i\dfrac{p(s)}{q(s)}$	> 0
t^n	$\dfrac{(-i)^{n+1}n!\,p(s)}{[q(s)]^{n+1}}$	> 0
$e^{\alpha t}$	$-p(s)\left[\dfrac{\alpha}{\alpha^2+\left(q(s)\right)^2}+i\dfrac{q(s)}{\alpha^2\left(q(s)\right)^2}\right]$	$q(s) > \alpha$
$\sin(\alpha t)$	$\dfrac{-\alpha p(s)}{\left(q(s)\right)^2-\alpha^2}$	$q(s) > \lvert\alpha\rvert$
$\cos(\alpha t)$	$\dfrac{-ip(s)q(s)}{\left(q(s)\right)^2-\alpha^2}$	$q(s) > \lvert\alpha\rvert$

If $T_g^c\{g(t)\} = F_g^c(s)$ is the new general complex integral transform, then the inverse would be

$$g(t) = T_g^{c^{-1}}\left[F_g^c(s)\right] = \frac{1}{2\pi}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{p(s)}e^{iq(s)t}F_g^c(s)\,ds.$$

Table 3 shows the inverse of SEJI integral transform for some elementary functions.

Table 3 Inverse functions of SEJI integral transform

Inverse function	Original function	Condition
$-i\dfrac{p(s)}{q(s)}$	1	$q(s) > 0$
$-\dfrac{p(s)}{[q(s)]^2}$	t	$n > 0$
$(-i)^{n+1}\dfrac{p(s)}{\left(q(s)\right)^{n+1}}$	$\dfrac{t^n}{n!}$	$n > 0$
$(-i)^{n+1}\dfrac{p(s)}{\left(q(s)\right)^{n+1}}$	$\dfrac{t^n}{\Gamma(n+1)}$	$n > -1$
$-p(s)\left[\dfrac{\alpha}{\alpha^2+\left(q(s)\right)^2}+\dfrac{iq(s)}{\alpha^2+\left(q(s)\right)^2}\right]$	$e^{\alpha t}$	$q(s) > \alpha$
$-\beta\dfrac{p(s)}{\left(q(s)\right)^2-\beta^2}$	$sin(\beta t)$	$q(s) > \lvert\beta\rvert$
$-i\dfrac{p(s)q(s)}{\left(q(s)\right)^2+\beta^2}$	$\cosh(\beta t)$	$q(s) > 0$

3. Result and Discussion

In this section, the solution of DTM is shown solving the ODE with variable coefficient referred from [6]. After that, analysis is conducted numerically and graphically.

3.1 Example 1

Consider the following second-order ODE [6]

$$y'' + xy' - y = 0 \tag{3}$$

with the initial condition,

$$y'(0) = 0, y''(0) = 1 \tag{4}$$

and given the exact solution,

$$y(x) = x. \tag{5}$$

By applying DTM, from the initial conditions and theorem, we obtain

$$Y(0) = 0,$$
$$Y(1) = 1.$$

The transformed equation would be

$$(k+2)(k+1)Y(k+2) + \sum_{r=0}^{k} \delta(r-1)(k-r+1)Y(k-r+1) - Y(k) = 0$$

and the final recurrence relation is

$$Y(k+2) = \frac{1-k}{(k+1)(k+2)} Y(k). \tag{6}$$

Substituting k=0,1,2,3,4, combining all terms would be

$$\boldsymbol{Y(x) = x.} \tag{7}$$

By using software Maple 2015, numerical solution for was obtained as shown in Table 4.

Table 4 Numerical analysis for Example 1

x	SEJI integral transform	DTM	Absolute error
0.0	0.0000000000	0.0000000000	0.00E+00
0.1	0.1000000000	0.1000000000	0.00E+00
0.2	0.2000000000	0.2000000000	0.00E+00
0.3	0.3000000000	0.3000000000	0.00E+00
0.4	0.4000000000	0.4000000000	0.00E+00
0.5	0.5000000000	0.5000000000	0.00E+00
0.6	0.6000000000	0.6000000000	0.00E+00
0.7	0.7000000000	0.7000000000	0.00E+00
0.8	0.8000000000	0.8000000000	0.00E+00
0.9	0.9000000000	0.9000000000	0.00E+00
1.0	1.0000000000	1.0000000000	0.00E+00

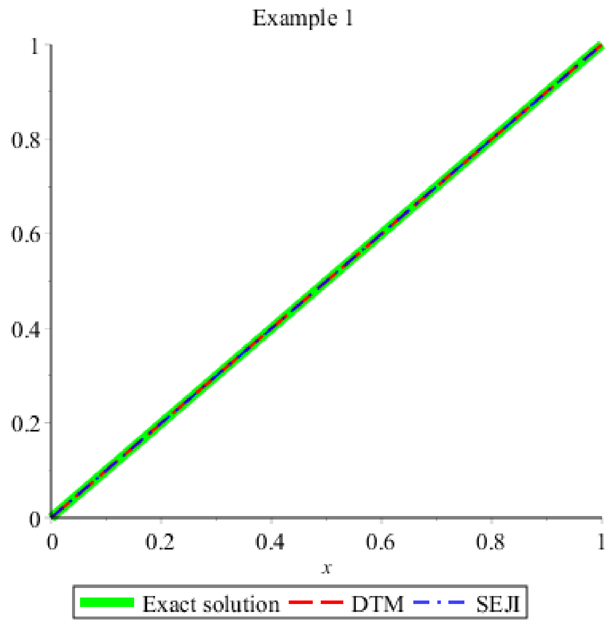

Fig. 1 Graphical analysis of exact, DTM and SEJI transform of Example 1

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Table 4 shows that both have all absolute errors of 0.0000000000. SEJI integral transform being analytical, we can simply deduce that the method will produce exact solution. Similarly, in Fig. 1, all the three lines overlap, suggesting that both methods produce solutions exactly like to the exact solution. While the graph and the numerical analysis provide an overall impression of similarity, it does not allow for a detailed evaluation of the accuracy and reliability of each method. To draw a more realistic conclusion, we need to try out different ODEs, to confirm whether similar case will occur.

3.2 Example 2

Consider the following second-order ODE [6]

$$xy'' + (1 - 2x)y' - 2y = 0 \tag{8}$$

with initial condition

$$y(0) = 1, y'(0) = 2 \tag{9}$$

and given the exact solution

$$y(x) = e^{2x}. \tag{10}$$

By applying DTM, from the initial conditions and theorem, we obtain

$$\mathrm{Y}(0) = 1,$$
$$\mathrm{Y}(1) = 2.$$

For the ease of calculation, the ODE is expanded.

$$xy" + y' - 2xy' - 2y = 0$$

The transformed equation will be

$$k(k+1)Y(k+1) + (k+1)Y(k+1) - \sum_{r=0}^{k} 2\delta(r-1)(k-r+1)Y(k-r+1) - 2Y(k) = 0$$

and the final recurrence relation is

$$Y(k+1) = \frac{2Y(k)}{k+1}. \tag{11}$$

Substituting up to k=5, combining all terms, the series is

$$Y(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}\, x^7. \tag{12}$$

By using software Maple 2015, numerical solution for was obtained as shown in Table 5.

Table 5 Numerical analysis for Example 2

x	SEJI integral transform	DTM	Absolute error
0.0	1.0000000000	1.0000000000	0.00E+00
0.1	1.221402758	1.221402667	9.10E-08
0.2	1.491824698	1.491818667	6.03E-06
0.3	1.8221188000	1.822048000	7.08E-05
0.4	2.225540928	2.225130667	4.10E-04
0.5	2.718281828	2.716666667	1.62E-03
0.6	3.320116923	3.315136000	4.98E-03
0.7	4.055199967	4.042218667	1.30E-02
0.8	4.953032424	4.923114667	2.99E-02
0.9	6.049647464	5.986864000	6.28E-02
1.0	7.3890560990	7.266666667	1.22E-01

Table 5 compares the exact solution, the DTM solution, and their absolute error across the interval $0 \le x \le 1$. Observations show that the DTM closely approximates the exact solution, with minimal errors for smaller values of x. However, as x increases, the absolute error gradually grows.

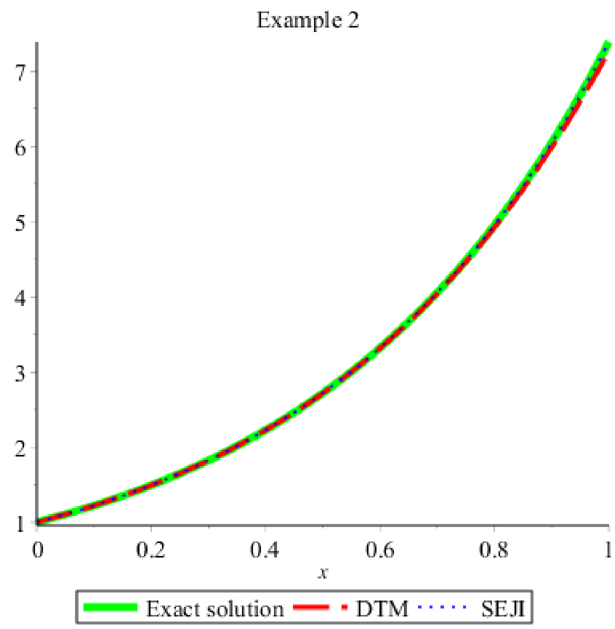

Fig. 2 Graphical analysis of exact, DTM and SEJI transform of Example 2

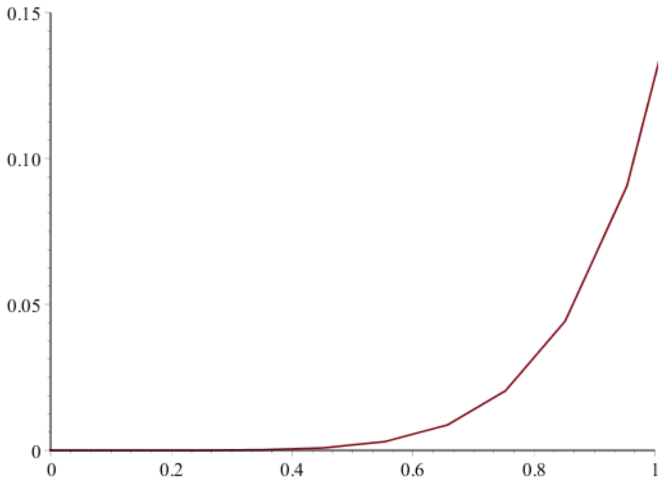

Fig. 3 Graphical analysis of absolute error of DTM for Example 2

Fig. 2 shows that the DTM closely approximates the exact solution, with minimal errors for smaller values of x, which is similar in the analysis in Table 5. However, as x increases, the absolute error gradually grows, reaching its highest value at $x = 1.0$. Despite this, the overall error remains relatively small, as displayed in Fig. 3, indicating that the DTM is nearly accurate for most of the interval. These results suggest that the DTM is a reliable method for solving differential equations, particularly for intervals where x is not too large.

4. Discussion

In this section, the similarities and differences between DTM and SEJI integral transform will be discussed.

4.1 Similarities

Both the DTM and the SEJI integral transform are primarily used to solve ordinary and partial differential equations by converting complex equations into simpler forms that are easier to manipulate and solve. Besides that, both methods transform functions into a new domain, with DTM converting them into a power series form, while SEJI applies a specific integral transformation. Moreover, they share key mathematical properties, including linearity, change of scale, time-shifting, and the convolution theorem. Additionally, both methods are applicable to initial value problems, enabling solutions based on specified initial conditions. Finally, they are also versatile and can be applied to various types of differential equations, including linear and delay differential equations, making them valuable tools in mathematical modelling and engineering applications.

4.2 Differences

The key difference is that DTM is semi-analytical method while SEJI integral transform is analytical method. Therefore, their differences are heavily influenced by the category of the method (analytical, numerical, and semi-analytical). Here we will discuss their comparison in terms of precision, complexity handling, computational efficiency, and application scope.

Firstly, in terms of precision, DTM provide approximate solutions that are not exact but often achieve higher accuracy than purely numerical methods. These methods combine analytical and numerical components to strike a balance between precision and practicality. On the other hand, SEJI integral transform yield exact solutions that are definitive and precise. This makes them ideal for problems where closed-form solutions are possible, offering unparalleled accuracy for well-defined cases.

Next, regarding complexity handling, DTM excels in handling complex problems, particularly those involving nonlinear equations or systems. They are versatile and capable of addressing challenges where traditional analytical methods fail by incorporating numerical techniques to refine solutions. In contrast, SEJI integral transform are best suited for simpler problems or those with specific structures that allow for closed-form solutions. However, they struggle with highly nonlinear or intricate systems due to the limitations of traditional mathematical frameworks.

When it comes to computational efficiency, DTM is generally more efficient for solving complex problems. It allows quicker approximations by deriving an initial analytical solution and refining it numerically, reducing the computational effort. Conversely, SEJI integral transform may demand significant time and effort to derive solutions for complicated equations. This can result in impractical computation times, especially when dealing with large-scale or highly intricate problems.

Finally, DTM are widely applicable across various disciplines, particularly in engineering and applied sciences. They are suitable for scenarios where exact solutions are not necessary, but a reliable approximation is beneficial. SEJI integral transform, however, are limited to problems where exact solutions are feasible. They are highly effective in theoretical contexts or when absolute precision is critical, but their applicability diminishes for more complex or real-world problems.

5. Conclusion

Both the differential transform method (DTM) and the SEJI integral transform serve as powerful tools for solving ordinary and partial differential equations by transforming complex equations into more manageable forms. Their shared mathematical properties, such as linearity, change of scale, time-shifting, and the convolution theorem, make them valuable in mathematical modelling and engineering applications. Additionally, their ability to handle initial value problems and different types of differential equations highlights their versatility in various scientific fields. Overall, the choice between DTM and SEJI depends on the specific problem at hand. If high accuracy with exact solutions is necessary, SEJI is the preferable method. However, if computational efficiency and the ability to handle complex, nonlinear equations are the priority, DTM offers a more practical and adaptable solution. By understanding the strengths and limitations of both methods, researchers can make informed decisions when selecting the most suitable approach for solving differential equations.

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Conflict of Interest

Authors declare that there is no conflict of interest regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Nur Sabrina Nasron and Noor Azliza Abd Latif;* ***data collection:*** *Nur Sabrina Nasron;* ***analysis and interpretation of results:*** *Nur Sabrina Nasron and Noor Azliza Abd Latif;* ***draft manuscript preparation:*** *Nur Sabrina Nasron. Noor Azliza Abd Latif, Norziha Che Him; All authors reviewed the results and approved the final version of the manuscript.*

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