ENHANCED KNOWLEDGE IN SCIENCES AND TECHNOLOGY
e-ISSN: 2773-6385

EKST

Vol. 4 No. 2 (2024) 103-109
https://publisher.uthm.edu.my/periodicals/index.php/ekst

Solving Newton's Law of Cooling Model using Euler Method, Fourth Order Runge-Kutta Method and Fourth Order Implicit Multistep Method

Nurul Syahirah Samat[1], Norzuria Ibrahim[1]*,

[1] *Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh, Hab Pendidikan Tinggi Pagoh, KM1, Jalan Panchor, 84600 Pagoh, Muar, Johor, MALAYSIA*

*Corresponding Author: norzuria@uthm.edu.my
DOI: https://doi.org/10.30880/ekst.2024.04.02.012

Article Info

Received: 27 December 2023
Accepted: 11 January 2024
Available online: 12 December 2024

Keywords

Newton's law of cooling, Ordinary differential equation, Runge-Kutta fourth order method, Euler Method, Numerical method error Analysis, MATLAB

Abstract

This paper discussed the numerical solution of first-order linear ordinary differential equations (ODEs) of newton's Law of Cooling (NLOC). NLOC stated that the rate of heat exchange between an object and its environment is proportional to the temperature difference between the object and the environment. Three proposed methods have been used are Euler method, fourth order Runge-Kutta method (4RKM) and fourth order Implicit Multistep method (4IMM). This research were done because generally analytic solution was hard to use for some complex problems and there are certain results that unable to obtain from the analytical method. The numerical method chosen for solving differential equation is solved using MATLAB and the results obtained were compared to obtained the most accurate methods. This research approximation methods use varying step sizes to compared with the exact solution. Two step sizes are chosen which are 0.1 and 0.05. The three numerical methods have been compared numerically and illustrated graphically. It was observed that 4RKM performs better than Euler method and 4IMM method in solving NLOC.

1. Introduction

Farlow (1994) stated that the rate of cooling of an object in a medium is described by Newton's law of cooling (NLOC). The rate of change in an object's temperature is related to the difference between that object's temperature and the surrounding air temperature using a first-order linear ordinary differential equation (ODEs). NLOC allowed us to refine, validate and extend our understanding of heat transfer processes, leading to practical applications and advancements in scientific knowledge. Applying analytic solution to the NLOC equation can provide an exact solution. However, analytic solution is hard to use for complex problems. Numerical methods enable us to solve complex problems and good results that analytic unable to achieve [1]. This study chooses three different numerical methods to solve the NLOC equation, which is Euler method, fourth order Runge-Kutta method (4RKM) and fourth order Implicit multistep Method (4IMM). Euler method is chosen because it is powerful and faster method for solving initial value problem. 4RKM is commonly used by mathematician to solve ODEs because it is more accurate for solving initial value problem of ODEs. Runge-Kutta method can be used to construct high order accurate numerical method by function's self without needing the high order derivatives of function.

This research aims on solving NLOC. This model commonly uses in professional fields such as biology, physics, and economics. This research also focused on solving NLOC using numerical method that have been chosen which is Euler method, 4RKM and 4IMM. MATLAB will be used to solve the NLOC equation.

There are two main classes of differential equation which is ODEs and partial differential equations (PDEs). So, ODEs are a differential equation which involves derivatives with respect to a single independent variable. While PDEs is a differential equation that contain two or more independent variables and partial derivatives with respect to them. In this case NLOC is ODE's. There are few types of ODEs which are linear and non-linear ODEs [2].

There are two main methods to solve ODEs numerically which is one-step method and multistep method. Example for one-step method is Euler method and example for multistep method is Adams-Bashforth method and Adams-Moulton methods. There are two type of multistep method which is explicit and implicit. In this study Implicit multistep method is chosen which is Adams-Moulton method. According to Faires & Burden (2016), the multistep methods approach more than one preceding point [7]. This research is investigated because numerical methods vary in their accuracy and stability under difficult conditions. Researching on application of Euler method, 4RKM and 4IMM allowed for a detailed analysis of their performances

Euler method is a first-order numerical procedure for solving ODEs with a given initial value. It is the most basic explicit method for numerical integration of ODEs and is the simplest Runge-Kutta method. The Euler method is named after Leonhard Euler. Euler method is a first order method, which means that the local error is proportional to the square of the step size. The Euler method often serves as the basis to construct more complex methods. It is a first-order numerical routine for solving ODEs with a given initial value and a fixed time step [3]. Euler's method is the simplest and the most fundamental numerical method for solving the first order ODEs [4].

4RKM is one of a numerical approach that used to solve the first order of ODEs. Euler Method and Runge-Kutta methods are used to solve higher ODEs or coupled differential equations. This method used slope approximations to estimate the slope at some time [5]. The author discussed that Runge-Kutta techniques were introduced around 1900 by Carl Runge and Martin Willhelm Kutta. After passage of time this method took a major role in the study of iterative methods based on explicit and implicit which applied to solve ODEs through temporal discretization.

2. Methodology

This research proposed to solve the NLOC using Euler's Method, 4RKM and 4IMM and then compared with exact solution. The comparison between these three methods was to see the accuracy of the method. In this study, an experiment of NLOC was conducted by taking a temperature of stew cheesecake bought from convenient store. The stew cheesecake was heated in a microwave with medium heat for 3 minutes and the initial temperature was taken after that. This experiment was demonstrated in room of 29 degree Celsius to get the constant of the temperature and the initial temperature. The experiment temperature was taken using the food thermometer. In this study it is assumed that the ambient temperature remains constant after 120 minutes.

Fig. 1 Experiment of NLOC

The experiment data is stored from 0 minutes until 120 minutes. Each temperature for all minutes were used for calculating the constant. The constant was calculated using the general exact solution of NLOC.

$$\frac{dT}{dt}(t) = -K(T(t) - M) \tag{1}$$

$$K = -ln\left[\frac{T(t)-M}{T(i)-M}\right]/t \quad (2)$$

$$T(t) = M + (T_i - M)e^{-Kt} \quad (3)$$

Equation (1) is the general solution of NLOC, equation (2) is the reformatted of equation (1) and used to find the constant of temperature rate of change. While equation (3) is used to find the exact solution that need to be compared with Euler method, 4RKM and 4IMM. The constant was calculated with equation (2), and we get *K=0.0389*. Since the initial temperature is *98.9℃* and the ambient temperature is *29℃* the exact solution is as below. Exact solution in Table 1 was used to compared with others approximation methods results Euler method, 4RKM and 4IMM using two different step sizes which is 0.1 and 0.05. The Euler method combines the first two term of the Taylor series with the differential equation to producing a stepping rule for solving the equation. Starting from any time step with a temperature, T at time, t. So, for each time step it is

$$T_{i+1} = T_i - kh(Ti - M) \quad (4)$$

For 4RKM, the method provided the approximate value of y for a given point x. Only the first order ODE's can be solved using the 4RKM. This method will give higher accuracy without performing more calculations. NLOC using 4RKM is shown as below.

$$k_1 = h[-K(T_i - M)] \quad (5)$$

$$k_2 = h[-K(T_i + 0.5k_1) - M] \quad (6)$$

$$k_3 = h[-K(T_i + 0.5k_2) - M] \quad (7)$$

$$k_4 = h\,[-K(T_i + k_3) - M] \quad (8)$$

To find the temperature for each time step using 4RKM will be

$$T_{i+1} = T_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (9)$$

Next numerical approach is 4IMM. It was also known as the implicit three-step method [8]. Multistep method approaches used the approximation at more than one prior mesh points to determine the approximation at the next points.

$$T_{i+1} = T_i + \frac{h}{24}\big[55\big(f(T_i, t_i)\big)\big] - 59\big[\big(f(T_{i-1}, t_{i-1})\big)\big] + 37\big(f(T_{i-2}, t_{i-2})\big) \\ -9[(f(T_{i-3}, t_{i-3}))] \quad (10)$$

$$T_{i+1} = T_i + \frac{h}{24}\big[9\big(f(T_{i+1}, t_{i+1})\big)\big] + 19\big[\big(f(T_i, t_i)\big)\big] - 5\big(f(T_{i-1}, t_{i-1})\big) \\ +\big(f(T_{i-2}, t_{i-2})\big) \quad (11)$$

The Implicit multistep method that will be used in solving the NLOC is Adam-Bashforth method or predictor for 4IMM as shown in equation (10) while in equation (11) shows Adam-Moulton method or corrector for 4IMM. All method were used in MATLAB to produce the result and compared with exact solution to assess the error and accuracy.

3. Results and Discussion

The food thermometer was used to measure the cheesecake temperature from 0 minutes until 120 minutes and the data recorded was used to find the constant and initial temperature. Based on this experiment the constant calculated is 0.0389 and the initial temperature is 98.9°C. After knowing the constant and the initial temperature, substitute it into equation (3) and recorded it as exact solution that need to be compared with numerical approach solution. The exact solution is shown as in table 1. The exact solution of the temperature rate of change that has been calculated is compared with approximate method result. The problem is tested with two different step sizes, *h* to determine its accuracy. The step sizes that have been considered in this study is 0.1 and to ensure the optimal accuracy it will be compared with another smaller step size which is 0.05. In this study the step size, *h* is also can be considered as time step. For h=0.1 means the temperatures were taken at every 0.1 minutes or every 6 seconds. The smaller step sizes compared is 0.05 means the temperatures were taken every 3 seconds. The results compared will only be in a range of 0 minutes until 120 minutes.

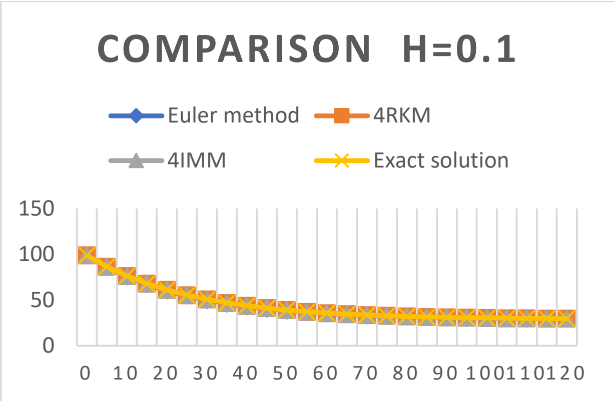

Fig. 2 Results for comparison h=0.1

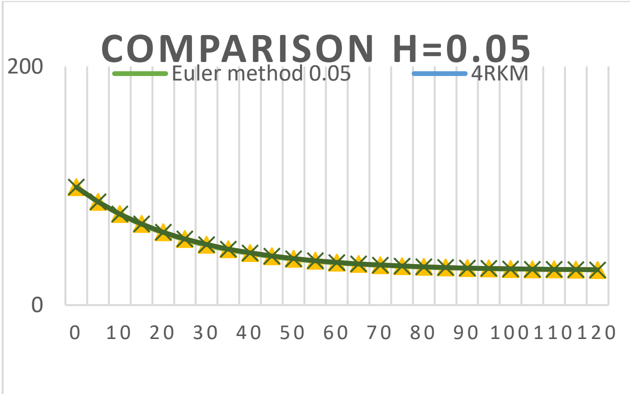

Fig. 3 Results for comparison h=0.05

The outcomes for the comparison of exact solution and Euler method, 4RKM and 4IMM that had been obtained is demonstrated in Fig. 2 and Fig. 3 with different step sizes, h=0.1 and *h=0.05.* Meanwhile error for the outcomes also illustrated in graph as in Fig. 4 and Fig. 5. The numerical results in fig 4 shows the results after compared the exact solution and numerical approach solution, which is Euler method, 4RKM and 4IMM with step sizes 0.1 minutes or 6 seconds. In other hand, fig 5 showed the results of comparison for smaller step sizes 0.05 minutes or 3 seconds. In this study it showed that there are not much different in exact solution and predicted numerical approach solution. However, absolute error for both step sizes were demonstrated in Fig. 4 and Fig. 5. The error for 0.1 minutes step sizes show that 4RKM has the most accurate results than the other two methods which is Euler method and 4IMM. While for error 0.05 minutes also shows that 4RKM has the most accurate results compared to Euler method and 4IMM.

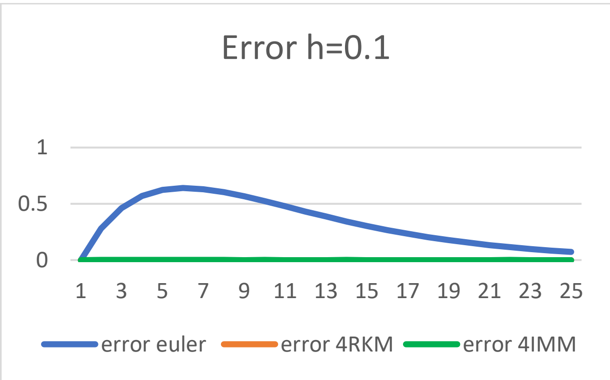

Fig. 4 Graphically results of error for h= 0.1

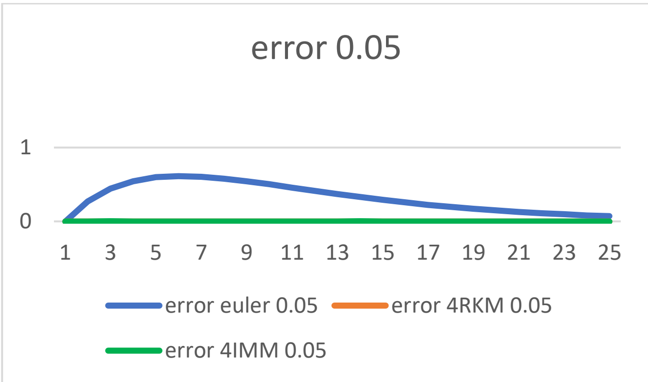

Fig. 5 Graphically results of error for h= 0.05

For every result from Fig. 2 and Fig. 3, it shows that an approximation solution is near to exact solution because if the time steps go to 0 the error also tend to 0. This study observed that the temperature is inversely proportional to the cooling time. If the time increases the temperature decreases. When the cheesecake is exposed outside the time taken for the cheesecake to be in thermal equilibrium with surroundings temperature of 29°C approximately 185 minutes.

Finally, this study observed that the 4RKM method is the most accurate method for solving initial value problem of Newton's Law. 4RKM formula appears to be complicated, however it produces very accurate approximations even when the number of iterations is reasonably small. The results obtained with numerical methods and exact solution proved that numerical method can be accurate in solving NLOC problems.

By referring to figure above, generally the aim for comparing three method and observed which numerical methods is more accurate had been achieved. It shows that the 4RKM methods give better approximations to the exact solution. By comparing the errors, it was discovered that the 4RKM at both step sizes give the less error. Referring to Fig 6 and Fig 7 4RKM results were overlapping with the exact solution. Finally, it is observed that the 4RKM method converging faster than Euler methods and it is the most accurate method for solving initial value problems for NLOC models and other first ODEs model.

4. Conclusions

This study focuses on solving NLOC model using numerical method approximation, which is Euler method, 4RKM and 4IMM and the results of approximation method were compared with exact solution that obtain. The exact solution is calculated after obtaining the constant value from experiment of NLOC. The exact solution is used to compared with the selected numerical methods using MATLAB software with various step sizes. In this research the 4RKM methods perform better than Euler method and 4IMM in terms of accuracy. From the findings 4RKM quickly approaches the exact solution answer. The error for 4RKM in this study were 0 all the time and clearly shown that the methods were very accurate to exact solution. Finally, it can be concluded that 4RKM is way better in solving NLOC problems compared to Euler method and 4IMM because it has the less error.

There are several recommendations that can be made for the future and research on the solving NLOC using Euler method, 4RKM and 4IMM. The NLOC model can be supplemented with a better experiment in a constant room temperature to avoid any error for calculating the constant value. Besides that, the research can also be done using other software such as python. It is because the software is widely used and easy to learn. In addition, this NLOC model can be further advanced by comparing the real-world data of temperature rate of change such as corpse temperature after death.

Acknowledgement

The authors would thank the Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia for its support.

Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design****: Nurul Syahirah Binti Samat:* ***Analyzed and interpreted data****. Nurul Syahirah Binti Samat and Norzuria binti Ibrahim. All authors reviewed the results and approved the final version of the manuscript*

Appendix A: Table For experiment and exact solution data

Time (minutes)	Experiment Temperature (°C)	Exact Solution (°C)
0	98.9	98.9
5	75.9	86.545
10	62.5	76.374
15	57	68
20	54.1	61.107
25	47.8	55.432
30	45.1	50.76
35	42.6	46.914
40	41.1	43.747
45	39.5	41.141
50	37.3	38.995
55	36.5	37.228
60	35.3	35.774
65	33.5	34.577
70	32.6	33.591

75	32	32.779
80	31.6	32.111
85	31.5	31.561
90	31.3	31.109
95	31	30.736
100	30.8	30.429
105	30.6	30.177
110	29.8	29.969
115	29.8	29.797
120	29.7	29.656

Appendix B: Table for comparison

Time (minutes)	Euler method 0.1	4RKM 0.1	4IMM 0.1	Euler method 0.05	4RKM 0.05	4IMM 0.05
0	98.9	98.9	98.9	98.9	98.9	98.9
5	86.264	86.545	86.544	86.275	86.545	86.545
10	75.912	76.374	76.373	75.931	76.374	76.373
15	67.431	68	67.999	67.454	68	68
20	60.484	61.107	61.106	60.509	61.107	61.107
25	54.792	55.432	55.431	54.818	55.432	55.432
30	50.13	50.76	50.759	50.155	50.76	50.76
35	46.31	46.914	46.913	46.334	46.914	46.914
40	43.181	43.747	43.747	43.203	43.747	43.747
45	40.617	41.141	41.14	40.638	41.141	41.141
50	38.517	38.995	38.995	38.536	38.995	38.995
55	36.797	37.228	37.228	36.814	37.228	37.228
60	35.387	35.774	35.774	35.402	35.774	35.774
65	34.233	34.577	34.576	34.246	34.577	34.576
70	33.287	33.591	33.591	33.299	33.591	33.591
75	32.512	32.779	32.779	32.522	32.779	32.779
80	31.877	32.111	32.111	31.886	32.111	32.111
85	31.357	31.561	31.561	31.365	31.561	31.561
90	30.931	31.109	31.109	30.938	31.109	31.109
95	30.582	30.736	30.736	30.588	30.736	30.736
100	30.296	30.429	30.429	30.301	30.429	30.429
105	30.062	30.177	30.176	30.066	30.177	30.177
110	29.87	29.969	29.969	29.873	29.969	29.969
115	29.712	29.797	29.797	29.716	29.797	29.797
120	29.584	29.656	29.656	29.586	29.656	29.656

Appendix C: Table for error

Time (minutes)	Euler 0.1	4RKM 0.1	4IMM 0.1	Euler 0.05	4RKM 0.05	4IMM 0.05
0	98.9	98.9	98.9	0	0	0
5	86.264	86.545	86.544	0.27	0	0
10	75.912	76.374	76.373	0.443	0	0.001
15	67.431	68	67.999	0.546	0	0
20	60.484	61.107	61.106	0.598	0	0
25	54.792	55.432	55.431	0.614	0	0
30	50.13	50.76	50.759	0.605	0	0
35	46.31	46.914	46.913	0.58	0	0
40	43.181	43.747	43.747	0.544	0	0
45	40.617	41.141	41.14	0.503	0	0
50	38.517	38.995	38.995	0.459	0	0
55	36.797	37.228	37.228	0.414	0	0
60	35.387	35.774	35.774	0.372	0	0
65	34.233	34.577	34.576	0.331	0	0.001
70	33.287	33.591	33.591	0.292	0	0
75	32.512	32.779	32.779	0.257	0	0
80	31.877	32.111	32.111	0.225	0	0
85	31.357	31.561	31.561	0.196	0	0
90	30.931	31.109	31.109	0.171	0	0
95	30.582	30.736	30.736	0.148	0	0
100	30.296	30.429	30.429	0.128	0	0
105	30.062	30.177	30.176	0.111	0	0
110	29.87	29.969	29.969	0.096	0	0
115	29.712	29.797	29.797	0.081	0	0
120	29.584	29.656	29.656	0.07	0	0

References

[1] Farlow, S. J. (1994). An introduction to differential equations and their application.
[2] Ahsan, Z. (2016). Differential Equation and Their Applications. Delhi : PHI Learning Private Limited
[3] Efimov, D., Polyakov, a., & Aleksandrov, A. (2019). Discretization of homogeneous systems using Euler method with a state-dependent step. Automatica.
[4] Nurujjaman, M. (2020). Enhanced Euler's Method to Solve First Order Ordinary . Journal of Engineering Mathematics & Statistics
[5] Chauhana , V., & Srivastava, P. K. (2019). Computational Techniques Based on Runge-Kutta Method of Various. International Journal of Mathematical, Engineering and Management Sciences, 13
[6] Tan, D., & Cheng, Z. (2012). A General Formula of Fourth Order Runge-Kutta. Journal of Mathematical Sciences & Mathematics Education.
[7] Fathoni, M. F., & Wuryandari, A. i. (2015). Comparison between Euler, Heun, Runge-Kutta and Adams-Bashforth-Moulton integration methods in the particle dynamic simulation. International Conference on Interactive Digital Media (ICIDM).
[8] Haizad Hakimi (2023) Solving Lotka-Volterra Prey Predator Model Using Implicit Multistep Method Of Order Four. Faculty of Applied Science and Technology Universiti Tun Hussein Onn Malaysia.