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A Numerical Methods to Obtain Exact Confidence Interval for Likelihood-Based Parameter Estimator

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Abstract

This study compares three distinct approaches for determining confidence intervals—the Bootstrap Method, Monte Carlo Simulation, and Wald Confidence Interval—within the framework of likelihood-based parameter estimation. Extensive simulations and practical evaluations spanning a wide range of statistical models are used to thoroughly analyze each approach's strengths and limitations in obtaining confidence intervals for various parameter estimators. This research exposes the nuanced strengths, shortcomings, and practical appropriateness of these approaches by thorough examination, considering their performance across varying sample sizes, model complexity, and data distributions. The findings provide useful recommendations for academics and practitioners by providing light on the best confidence interval estimate approach for likelihood-based parameter estimation in real-world statistical settings.

1. Introduction

Because of its successful estimators, likelihood-based parameter estimation has gained traction in recent years [1]. However, defining confidence intervals for these estimators, which are critical for understanding parameter uncertainty, remains difficult, particularly with small samples or non-normal data [2]. The purpose of this study is to examine three strategies for improving the accuracy of confidence intervals: Bootstrap, Monte Carlo Simulation, and Wald Confidence Interval.

The primary objective of this study is to derive meaningful insights from the 'Poverty and Population Statistics by District in Negeri Sembilan' dataset from data.gov.my, employing likelihood-based parameter estimation techniques and comparing results through bootstrap, Monte Carlo simulations, and the Wald confidence interval, with a specific focus on informing policy choices and intervention activities tailored to the unique context of Negeri Sembilan.

The purpose of this study is to develop and evaluate three approaches for producing confidence intervals to determine the most accurate way for likelihood-based parameter estimate [3]. The study examines the precision and accuracy of confidence intervals produced from the Bootstrap, Monte Carlo Simulation, and Wald Confidence Interval methodologies in Negeri Sembilan, Malaysia. This study overcomes theoretical gaps and improves statistical reliability by improving confidence interval approaches, with important implications for decision-making in healthcare, economics, and social sciences [4].

2. Methodology

This chapter goes into using three numerical methods—the Bootstrap Method, Monte Carlo Simulation, and Wald Confidence Interval—to create correct confidence intervals for a likelihood-based parameter estimator. Our research will compare and evaluate various techniques using real-world data, measuring their correctness and applicability. The methodology portion includes data collection and preparation, followed by a full examination of each numerical approach and its real-world application. Our major objective is to identify the most efficient approach for creating exact confidence intervals.

2.1 Bootstrap method

Efron who introduced the bootstrap method was a game changer in statistical analysis, providing amazing flexibility and dependability in creating estimations without hard assumptions. Its adaptability has increased its utility in a variety of sectors. Davison and Hinkley 'further advances, which included bias corrections and acceleration algorithms, considerably increased the accuracy of confidence interval estimations [5]. Another significant advancement in the bootstrap method is the development of the parametric bootstrap technique by [6], which combines the strengths of both parametric and non-parametric methods by generating bootstrap samples from a tailored parametric model, thereby broadening the method's applications. Furthermore, [7]. Demonstrate the method's potential in increasing model performance through complete comprehension and treatment of uncertainties by using it to refine neural network training procedures. Recent advances, such as enhanced confidence interval estimations, the introduction of the parametric bootstrap, and its expanded applications, have solidified the bootstrap technique as a significant tool in finance, machine learning, statistics, and a variety of other fields.

2.2 Monte Carlo Simulations

Monte Carlo simulations, which use random sampling and iterative approaches, are essential for simulating complex systems across disciplines. This study focuses on recent advances in simulation efficiency and accuracy using variance reduction techniques such as Antithetic Variates and Control Variates [8]. These simulations are essential for statistical inference, estimating sample distributions, calculating standard errors, and evaluating model behavior. Recent research [9], demonstrate their relevance in investigating the behavior of non-parametric statistics and providing thorough model evaluation via unique selection criteria based on Monte Carlo simulations [10]. Overall, Monte Carlo simulations are essential for estimating parameters, exploring parameter spaces, and assessing statistical models, with recent advancements greatly improving their accuracy and efficiency.

2.3 Wald Confidence Interval

The Wald confidence interval is important in statistical inference because it relies on estimators' asymptotic normality to efficiently quantify parameter uncertainty. It is constructed by creating intervals using known variance estimators and the assumption of an asymptotic normal distribution, estimating margins of error from standard errors, and determining critical values for the standard normal distribution. Despite its simplicity, the Wald interval may fall short in situations with small sample sizes or biased estimators, resulting in wider intervals or insufficient coverage. To address these limits, other approaches like the likelihood ratio and bootstrap intervals are used. Understanding the Wald interval is linked to understanding the theoretical underpinnings of maximum likelihood estimation, as underlined by [11], who caution against depending entirely on it. The Wald interval is used in medical research, as demonstrated by [12], who evaluated a new medicine's effects on lowering blood pressure in hypertensive persons and found it to have a significant effect.

2.4 Data Collection And preparation

The study utilizes the dataset "Poverty and Population Statistics by District in Negeri Sembilan, Malaysia" from the Malaysian Government Open Data Portal (data.gov.my). This dataset includes key variables like "Year," "District," "Poverty," and "Extreme Poverty," allowing for an in-depth examination of poverty and population dynamics at the district level.

Collected by the Ministry of Federal Territories of Malaysia through surveys and censuses, the dataset likely employed systematic or stratified sampling to ensure representation across Negeri Sembilan's districts. Post-collection, the data underwent a thorough cleaning process, addressing missing values and outliers. Techniques such as imputation and exclusion were used for missing values, and outliers were treated to avoid undue influence.

The decision to use this dataset is based on its natural relevance to the study's focus. The project aims to gain insights into poverty dynamics in Negeri Sembilan, benefiting from the dataset's comprehensive information and meticulous collection methods.

2.5 Numerical Method to obtain Confidence Interval

2.5.1 Bootstrap Method

The study employs the Bootstrap Method to accurately estimate confidence intervals for assessing poverty and population trends in Negeri Sembilan, Malaysia. This resampling technique involves randomly selecting observations from the dataset, generating multiple bootstrap samples, and computing likelihood-based parameter estimators for each sample. The resulting bootstrap distribution, comprised of estimators from numerous resampling iterations, is then used to calculate confidence intervals. By determining the relevant quantiles based on the chosen confidence level, the method provides a robust framework for estimating the poverty rate in Negeri Sembilan, ensuring consistent and reliable insights into poverty and population dynamics in the region.

2.5.2 Monte Carlo Simulations

The Monte Carlo Simulation proves instrumental in precisely assessing poverty and population dynamics in Negeri Sembilan, Malaysia. This numerical method involves generating random samples from an assumed distribution or model to obtain accurate parameter estimates and confidence intervals. The number of simulations (N) is crucial, ensuring statistical validity and computational efficiency. Likelihood-based parameter estimators ($\hat{\theta}$) are computed for each simulated sample, capturing measures of interest like poverty rates. By conducting N simulations, a Monte Carlo distribution of parameter estimators ($\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N$) is generated, and confidence intervals are determined by calculating relevant quantiles. The resulting intervals, based on a specified confidence level, offer a credible range of values for the population parameter of interest, considering inherent variability in the simulated datasets. In conclusion, the Monte Carlo Simulation provides a robust numerical tool for obtaining precise estimates and confidence intervals, contributing valuable insights for informed decision-making and policy development regarding poverty and population dynamics in Negeri Sembilan.

2.5.3 Wald Confidence Interval

The Wald Confidence Interval method is employed for precise parameter estimation and confidence interval calculation in studying poverty and population trends in Negeri Sembilan, Malaysia. This approach relies on assumptions like asymptotic normality and unbiasedness. The confidence interval is determined using the formula CI = $\hat{\theta} \pm Z * SE$, where $\hat{\theta}$ is the parameter estimator, Z is the standard normal quantile associated with the desired confidence level, and SE is the standard error of the estimator, computed as the square root of the estimated variance. Maintaining assumptions of asymptotic normality and unbiasedness is crucial for accurate standard error estimation, as it allows the estimator to follow a normal distribution. Adjustments to the Wald interval may be made based on data or model characteristics, such as accommodating small sample sizes or incorporating robust standard errors for heteroscedasticity. The choice of the Wald Confidence Interval for this investigation is justified by its theoretical features, simplicity, interpretability, and asymptotic properties. However, it is emphasized that ensuring the fulfillment of underlying assumptions, especially asymptotic normality, is essential for trustworthy and valid findings.

2.6 Comparison And Analysis

The Comparison and Analysis section is pivotal in evaluating the accuracy and applicability of three numerical methods—Bootstrap, Monte Carlo Simulation, and Wald Confidence Interval—for likelihood-based parameter estimation in Negeri Sembilan, Malaysia. The primary focus is on comparing outcomes to determine which method yields more accurate results. Key performance indicators, including coverage probability and interval width, are calculated to assess accuracy. A higher coverage probability and narrower interval width signify a more accurate method, capturing the true parameter value with precision. Computational efficiency and robustness are also considered. By examining these metrics, a well-justified conclusion is drawn regarding the most accurate approach, facilitating informed judgments and recommendations regarding poverty and population dynamics in Negeri Sembilan.

3. Result and Discussion

This chapter delves into and compares three numerical strategies for estimating parameters based on probability in Python: Bootstrap, Monte Carlo simulations, and the Wald confidence interval. We're delving into a dataset organized by year and district to calculate confidence intervals for poverty and extreme poverty levels, with the goal of determining how exact and reliable these methodologies are. Using Python's statistical prowess, we're seeing how well various algorithms do at estimating the factors we're interested in, particularly poverty levels. The chapter has parts that break down the outcomes of the, which are followed by the three methods that were chosen. After that, we'll compare and debate what we've discovered. Sifting through poverty and extreme poverty patterns over several years. We're aiming to figure out the best approach to use statistics to generate confidence intervals, which is critical for estimating likelihood-based parameters.

3.1 Bootstrap Method Result

In this statistical analysis, the Bootstrap technique was employed to estimate confidence intervals for combined poverty and extreme poverty levels in Negeri Sembilan from 2014 to 2021. We can assume that the poverty as X_1 and extreme poverty as X_2. The datasets included:

X_1 = (1635, 3090, 2487, 2322, 2146, 2178, 2397, 6839),

X_2 = (268, 249, 31, 7, 4, 61, 251, 1047)

The Bootstrap method involved creating 1000 resampled datasets for each, simulating uncertainty by randomly selecting, with replacement, 8 elements from the original datasets. The mean was chosen as the statistic of interest, calculated as the sum of poverty or extreme poverty levels divided by the number of years.

Using Python and the NumPy library for efficiency, the analysis generated 1000 statistics through iterations, organized them in ascending order, and computed a 95% confidence interval by finding percentiles. The result provides a confident range for mean poverty and extreme poverty levels in Negeri Sembilan, offering valuable insights into the uncertainties and variations in poverty patterns from 2014 to 2021.

Table 1 Result from python of Bootstrap Method

Result	Poverty	Extreme Poverty
Mean	2886.75	239.75
Lower Bound	2122.1688	65.4812
Upper Bound	4024.9594	493.4344

According to the Bootstrap method study, the average poverty level in the Negeri Sembilan district from 2014 to 2021 was estimated at approximately 2886 people. This represents the average number of individuals experiencing poverty during this period. However, there is notable uncertainty in this estimate, as indicated by a wide confidence interval ranging from 2122 to 4024 persons. The lower limit (2122 people) represents the minimum conceivable average poverty level, while the upper limit (4024 people) signifies the most plausible average.

In comparison, the mean extreme poverty level was projected to be 239 people, indicating that the average number of individuals experiencing extreme poverty is lower than those experiencing ordinary poverty. Nevertheless, there is considerable variability in extreme poverty estimates, with a broad confidence range spanning from 65 to 493 people. This wide range underscores the substantial uncertainty associated with extreme poverty statistics, reflecting fluctuations in extreme poverty occurrences.

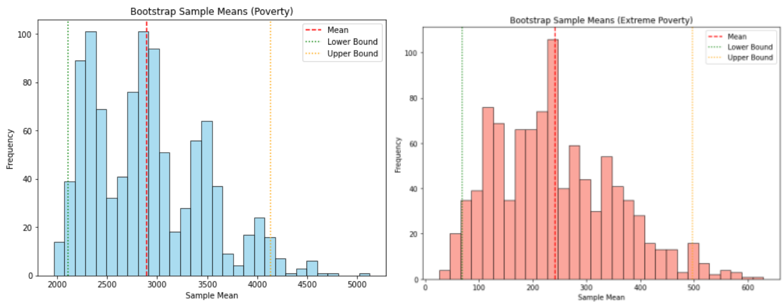

Fig. 1 Histogram graph of bootstrap method

The explanation of Bootstrap sample means for poverty and extreme poverty levels in Seremban is visually presented through two graphical representations. In the first graph, a dashed red line depicts the mean estimate, while dotted lines in green and orange indicate the lower and upper boundaries of the 95% confidence interval, respectively. The second graph, focusing on severe poverty levels, also features a dashed red line for the mean estimate, with dotted lines in green and orange representing the lower and upper boundaries of the 95% confidence interval. These graphics provide an intuitive insight into the distribution and confidence associated with the projected mean poverty and extreme poverty levels in Seremban.

3.2 Monte Carlo Simulation result

In this analysis, the Monte Carlo simulation, a powerful computational technique, is employed to estimate confidence intervals using the previous dataset (poverty and extreme poverty levels in the Seremban district of Negeri Sembilan from 2014 to 2021). Python, known for its simplicity and access to robust libraries, particularly NumPy for numerical calculations, is chosen as the programming language.

The objective is to determine confidence intervals for the mean values of poverty, X_1 and extreme poverty, X_2 based on a given dataset. The Monte Carlo simulation involves 1000 iterations, each simulating random sampling with replacement. In each iteration, random samples of poverty and extreme poverty levels are drawn, and the mean is calculated. This process is repeated, generating collections of sample means.

The mean value, obtained by averaging the sample means across iterations, serves as an estimated average level of poverty and extreme poverty in the Seremban area over the studied years. To quantify uncertainty, 95% confidence intervals are calculated by sorting the means and selecting the 2.5th and 97.5th percentiles.

Python, along with the NumPy library, facilitates efficient iterations and accurate calculations. The automated process of Monte Carlo simulations enhances the assessment of poverty dynamics, providing insights into the estimated mean levels and associated confidence intervals for both poverty and extreme poverty in the Seremban district. The results contribute valuable information to our understanding of poverty trends in the region, acknowledging the inherent uncertainty and variability in sampling methods. This approach ensures a comprehensive perspective on poverty dynamics, aiding decision-making processes and policy formulation.

Table 2 Result from python of Monte Carlo Simulation

Result	Poverty	Extreme Poverty
Mean	2902.29	238.22
Lower Bound	2108.83	71.60
Upper Bound	4046.64	490.02

For the estimated average poverty level in the Seremban area from 2014 to 2021, the confidence interval is provided as ranging from 2108 to 4046 individuals. This interval, calculated through Monte Carlo simulations, represents the range within which the true average poverty level is likely to fall with 95% confidence. The lower limit (2108 individuals) indicates the minimum plausible average poverty level, while the upper limit (4046 individuals) is considered the most likely average within the provided range.

Similarly, for the mean projected level of extreme poverty, the confidence interval is reported as ranging from 71 to 490 individuals. This wide interval signifies the uncertainty associated with extreme poverty data and reflects the potential variability in extreme poverty incidence over the study period. The lower limit (71 individuals) represents the minimum plausible average extreme poverty level, while the upper limit (490 individuals) is considered the most likely average within the provided range.

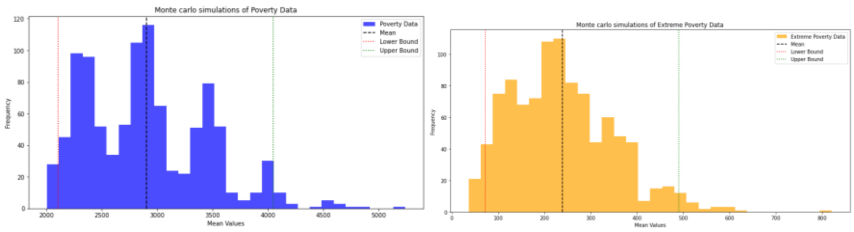

Fig. 2 Histogram graph of Monte Carlo Simulation

3.3 Wald Confidence Interval Result

In this research, the Wald confidence interval method was utilized to estimate population parameters for both poverty, X_1 and extreme poverty, X_2 levels in the Seremban district. Python, along with NumPy and SciPy, was employed for accurate and efficient statistical computations.

For the poverty dataset, X_1 with observations (1635, 3090, 2487, 2322, 2146, 2178, 2397, 6839), the mean was calculated as 2886.75. Similarly, for extreme poverty, X_2 with observations (268, 249, 31, 7, 4, 61, 251, 1047), the mean was found to be 239.75. The sample standard deviation, S was computed for both datasets, resulting in standard deviation for poverty is 1830.21 and standard deviation for extreme poverty = 362.89. The sample size for both datasets remained constant at 8.

The Wald confidence interval was constructed using the formulas of lower and upper bound and for a 95% confidence level, the critical value, Z was set at 1.96.

For Poverty X_1: Confidence interval is (1745.14, 4028.36) and for Extreme Poverty, X_2 Confidence interval (-0.01, 479.51)

Table 3 Result from python of Wald Confidence Interval

Result	Poverty	Extreme Poverty
Mean	2886.75	239.75
Lower Bound	1745.14	- 0.01
Upper Bound	4028.36	479.51

In this study, we used Python to generate careful Wald confidence intervals, which resulted in exact estimations of population parameters while reducing the chance of manual mistakes and automating difficult statistical procedures. This computational technique not only reduced time and effort, but it also regularly produced trustworthy results, increasing the overall dependability of our findings. Because these computational procedures are automated, they can handle bigger datasets more efficiently, allowing for more thorough statistical analysis than human methods. By using Python for statistical calculations, we not only produced correct results, but also dived further into data analytics, allowing for a more in-depth study of underlying insights and complex data patterns.

To graphically illustrate the estimated levels of poverty and extreme poverty in Negeri Sembilan, we used Wald confidence intervals in Python, together with the NumPy. The histograms show the distribution of mean estimates across both datasets. In the first histogram, a dashed black line indicates the mean estimate, which is surrounded by dotted red and green lines denoting the lower and upper boundaries of the 95% confidence range. The second histogram, which focuses on severe poverty, uses a similar design, with a dashed black line representing the mean estimate and dotted red and green lines outlining the lower and higher limits of the 95% confidence range.

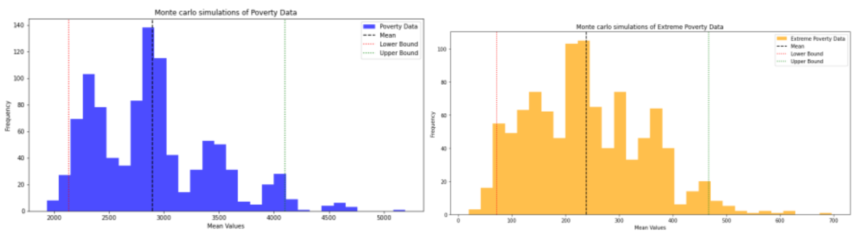

Fig. 3 Histogram graph of Wald Confidence Interval

3.4 Comparison and Analysis

The relevance of defining confidence intervals stems from understanding the potential range of values within which a real population parameter, the Bootstrap Method, Monte Carlo simulations, and Wald confidence intervals are three approaches for dealing with the inherent uncertainty in statistical estimates. Each method employs unique methodologies, such as producing bootstrap samples, deploying randomly created samples, or depending on exact assumptions about data distribution, to construct confidence intervals adapted to diverse data characteristics and analytical objectives.

The findings for both poverty and severe poverty datasets are examined using the Bootstrap, Monte Carlo, and Wald approaches, differences, and similarities in estimated mean values and their accompanying 95% confidence intervals become apparent. Poverty estimates were quite consistent across techniques, with averages of 2886.75 (Bootstrap), 2902.29 (Monte Carlo), and 2886.75 (Wald). However, the range of confidence intervals differed dramatically. While the Bootstrap and Monte Carlo techniques produced identical ranges of 2122.17 to 4024.96 and 2108.83 to 4046.64, respectively, the Wald confidence interval provided a larger range of 1745.14 to 4028.36.

Extreme poverty means estimates showed consistency, ranging from 238.22 to 239.75 across all methods. Nonetheless, confidence interval ranges varied significantly. The Bootstrap and Monte Carlo techniques yielded consistent intervals of 65.48 to 493.43 and 71.60 to 490.02, respectively. In comparison, the Wald confidence interval had a wider and clearer range, ranging from -0.01 to 479.51. These differences highlight the various levels of precision and accuracy in capturing the real population parameter within the datasets, depending on the approach used.

Bar plots graphically display mean estimations and associated 95% confidence intervals produced from statistical methods such as Bootstrap, Monte Carlo simulations, and Wald confidence intervals. The X-axis of these graphs is labelled with the techniques used (Bootstrap, Monte Carlo, and Wald), and vertical lines extend from each mark to illustrate the range of the matching confidence intervals that contain the mean estimations. These error bars effectively demonstrate the inherent variety and uncertainty in the projected mean values of both the poverty and extreme poverty datasets. The varying lengths of these bars represent the range of possible values, which capture statistical variations unique to each estimating approach. The visual depiction gives a clear picture of the distribution and confidence levels associated with mean estimations from various statistical methods.

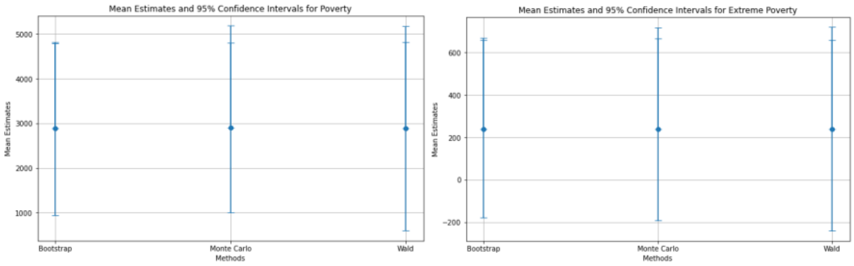

Fig. 4 Bar plot Graph Comparison

4. Conclusion

The use of likelihood-based parameter estimation methods (Bootstrap, Monte Carlo simulations, and Wald Confidence Intervals) to estimate poverty and extreme poverty levels in Negeri Sembilan yields useful results. Bootstrap, noted for its resilience, works well with a wide range of data but can be computationally intensive for bigger datasets. Monte Carlo simulations excel in modeling complexity, but they require many iterations. Wald Confidence Intervals, selected for their simplicity, are appropriate for bigger datasets but presuppose data normality.

The comparison findings reveal consistent mean poverty estimates, with Bootstrap and Monte Carlo approaches having higher precision and narrower confidence ranges. Wald Confidence Intervals produce larger ranges, particularly for extreme poverty, reflecting greater uncertainty. From 2014 to 2021, the estimated average poverty level is 2886.75 people, with 239.75 people experiencing extreme poverty. Wide confidence intervals emphasize inherent uncertainties, particularly in extreme poverty.

Method selection must consider dataset properties as well as trade-offs between efficiency and precision. Recognizing the relevance and limitations of the present work is critical. The larger confidence intervals reflect inherent uncertainty, prompting cautious interpretation, especially in extreme poverty estimations. Future study might improve approaches for computational problems and investigate models that accommodate non-normality, using qualitative data to provide a more complete picture of poverty dynamics in Negeri Sembilan.

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Conflict of Interest

The authors confirm that there are no conflicts of interest related to the publication of this paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design****: Aiman Imran Abd Rani;* ***solve the equations****: Aiman Imran Abd Rani;* ***analysis and interpretation of results****: Aiman Imran Abd Rani, Mahathir Mohamad;* ***draft manuscript preparation****: Aiman Imran Abd Rani, Mahathir Mohamad.*

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