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# Laplace Transform and Fourth Order Runge-Kutta Method for Solving Population Growth Models

**Goh Chin Yang[1], Syahirbanun Isa[1]***

[1] *Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology, UTHM Kampus Cawangan Pagoh, Hab Pendidikan Tinggi Pagoh, KM 1, Jalan Panchor, 84600 Pagoh, Muar, Johor, MALAYSIA*

*Corresponding Author: syahir@uthm.edu.my
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**Abstract**

There are two population growth models, which are the exponential model and logistic model in this project. These two population models are solved using Laplace transform and the fourth order Runge-Kutta method. Laplace transform is used directly to solve exponential model because it is a linear ordinary differential equation. Since the logistic model is a nonlinear ordinary differential equation, Adomian polynomials are applied to solve the problem by Laplace transform. The MATLAB software is used to calculate the solutions by the RK4 method. Two step sizes are considered, which are $h = 10$ and $h = 1$. The absolute errors are calculated, and the results are summarized in graphs to compare the results of solving population growth models using Laplace transform and RK4 method. The Laplace transform give better solution in exponential method, while the RK4 method give better solution in logistic model.

## 1. Introduction

Differential equations arise often in mathematical models which attempt to illustrate real-life situations. Many natural laws can be translated through mathematical language into equations related to derivatives. First order ordinary differential equations (ODEs) are widely used in applications like population growth models. There are many models of population, which involve linear ODEs and nonlinear ODEs, such as exponential model and logistic model [1]. Exponential population growth is related to Malthusian law of population growth which is named after Thomas Robert Malthus. Malthusian law of population growth is the rate of population proportional to the population present. Malthusian law is also sometimes referred to as the exponential law and it is widely regarded in the field of population ecology as the first principle of population dynamics [2]. The logistic model is a mathematical model that describes population growth, considering limiting factors and carrying capacity. It is widely used in population ecology and other fields to study the dynamics of populations over time. The logistic model is expressed as a nonlinear differential equation, where the rate of change of population size depends on the current population size, the growth rate, and the carrying capacity [3].

Population growth models are one of the applications of first order ODEs. The problems can be solved by analytical methods and numerical methods [1]. Laplace transform will be used to solve the problems analytically while RK4 method will be used to solve the problems numerically [2]. The numerical methods are well suited to the computer because the methods need a lot of time to solve the problems as more functions need to be applied [3]. Software like MATLAB is applied as a tool to solve problems numerically. The results of Laplace transform and RK4 method in solving population growth models will be compared.

Laplace transform is very useful and powerful in mathematics because it is a tool to solve differential equations. The Laplace transform is named after its discoverer Pierre-Simon, marquis de Laplace. Laplace's equation was formulated by Laplace, and Laplace pioneered the Laplace transform which is widely used in many branches of mathematical physics. The Laplace transform can be interpreted as a transformation from a function of a real variable in the time domain to a function of a complex variable, which is in the frequency domain [4].

The fourth order Runge-Kutta (RK4) method is one of numerical methods to solve ODEs problems. This method was originally developed by Carl David Runge and Martin Wilhelm Kutta. This method approximates exact solution and has smaller errors if compared to other numerical methods. This method is easy to apply, and it is sufficiently accurate to solve various problems productively. The RK4 method can be applied to solve population growth models because these problems involve first order ODEs [5].

The research focuses on solving population growth models, which are exponential model and logistic model. The exponential model is a linear ODE while the logistic model is a nonlinear ODE. Previous studies proposed several approaches in solving population growth problems. [6] presented application of Laplace transform for solving population growth and decay problems. The population growth is an exponential model. In [6], a general differential equation with initial condition is given and it can be solved. Linearity properties of Laplace transform, and Laplace transform of some basic functions are listed in the table. There are two examples of applications of Laplace transform. The difference between [6] and this study is [6] use Laplace transform to solve the population growth and decay problems, while this study use both Laplace transform and RK4 method to solve the exponential model and the logistic model.

The Laplace decomposition algorithm is applied in [7] to a class of nonlinear differential equations. This article provides a numerical Laplace transform method based on the technique of decomposition towards the approximate solution of a class of nonlinear differential equations. The method is explained and demonstrated using numerical examples.

Author in [8] introduced applications of Laplace transform in engineering fields. The definition and important properties of Laplace transform are mentioned. The author displayed methods to apply Laplace transform to solve differential equations. To better understand how to apply Laplace transform, there are some examples to solve the problem by Laplace transform.

RK4 method is one of accurate methods in solving ODEs. [9] proposed a comparative study on fourth order and Butcher's fifth order Runge-Kutta methods with third order initial value problem. This study explained the general equations of RK4 method and its initial value problem (IVP) for the system of three differential equations. It is similar to Butcher's fifth order Runge-Kutta method (RK5). Numerical approximations and maximum errors for different step sizes are calculated to compare RK4 method and RK5 method.

[10] illustrated the numerical accuracy of Runge-Kutta second, third and fourth order methods. The equations are solved by Runge-Kutta second, third and fourth order methods with the help of MATLAB. [11] presented the accuracy comparison of the RK4 and RK5 method of Susceptible-Exposed-Infected-Recovered (SEIR) model for tuberculosis cases in South Sulawesi. Numerical Solution of SEIR Models for TB in South Sulawesi by RK4 and RK5 are solved by using Maple and summarized in the tables.

The objectives of this study are to solve population growth models using the method of Laplace transform and RK4 method. This study also compares the results of solving population growth models using Laplace transform and RK4 method.

## 2. Materials and Methods

Population growth models are one of the applications of first order ODEs. For exponential model, based on Malthusian law of population growth, the rate of population is directly proportional to the population present, and it can be expressed in the following equation [2]:

$$\frac{dN(t)}{dt} = kN(t), \tag{1}$$

where $N(t)$ is the population of the given species at time $t$ and $k$ is the rate constant or growth constant. The initial value of time $t$ which is $t_0$ [2] is given by

$$N(t_0) = N_0. \tag{2}$$

In general, different starting values will lead to different solutions. The initial conditions will be fixed to obtain the unique solution.

The logistic model is a population growth model that considers limiting factors and carrying capacity. The logistic model can be expressed as follows [3]:

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right), \tag{3}$$

where $N(t)$ represents the population size at time $t$, $r$ is the growth rate of the population and $K$ is the carrying capacity which represents the maximum population size that the environment can sustain. The growth rate of the population, $r$ can be calculated by finding the difference between the birth rate and the death rate, and divided by the initial value [2]. The initial condition [3] is given by

$$N(t_0) = N_0, \tag{4}$$

where the initial value of time $t$ is $t_0$. The logistic model provides a more realistic representation of population growth compared to simple exponential models, which assume unrestricted growth [2].

## 2.1 Laplace Transform

The Laplace transform is a mathematical tool applied to solve linear ODEs. It transform linear ODEs into algebraic equations that are simpler to solve [4]. The Laplace transform changes a function of a real variable $t$ into a function of a complex variable $s$, which allows us to manipulate and analyze the function using algebraic operations rather than differential equations. Assume that $f(t)$ is a function for all $t \geq 0$, the Laplace transform of $f$, which is represented by $F(s)$ or $\mathcal{L}\{f(t)\}$ [4], is defined by the equation

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \tag{5}$$

with kernel $k(s,t) = e^{-st}$.

The Laplace transform of the derivative of a function $f(t)$ can be expressed using the $s$-variable transform, denoted as $\mathcal{L}\{f'(t)\}$, where $f'(t)$ represents the derivative of $f(t)$ with respect with $t$. The following formula express the Laplace transform of $n$th derivative of a function $f(t)$ [2]:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0), \tag{6}$$

where $f^{(n)}(t)$ is the $n$th derivative of $f(t)$ with respect to $t$, and $f(0), f'(0), \ldots, f^{(n-1)}(0)$ represent the initial value of the function and its derivatives at $t = 0$.

The inverse Laplace transform is an operation that changes a function or expression from the Laplace domain back to the time domain [3]. Given a function $F(s)$ in the Laplace domain, the inverse Laplace transform yields the corresponding function $f(t)$ in the time domain [3]. It is often represented as follows

$$f(t) = \mathcal{L}^{-1}\{F(s)\}. \tag{7}$$

Since Laplace transform is not suitable for directly solving nonlinear ODEs, the Adomian polynomials are applied in nonlinear differential equations. The Laplace transform contains next of representing the solution as an infinite series [7], namely

$$y = \sum_{n=0}^{\infty} y_n, \tag{8}$$

where the components $y_n$, $n \geq 0$ will be recursively computed. The nonlinear operator $f(y)$ is decomposed [7] in the form

$$f(y) = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where the so-called Adomian polynomials $A_n$ can be evaluated for all forms of nonlinearity. The first few polynomials [7] are given by

$$A_0 = f(y_0), \tag{10}$$

$$A_1 = y_1 f^{(1)}(y_0), \tag{11}$$

$$A_2 = y_2 f^{(1)}(y_0) + \frac{1}{2!} y_1^2 f^{(2)}(y_0), \tag{12}$$

$$A_3 = y_3 f^{(1)}(y_0) + y_1 y_2 f^{(2)}(y_0) + \frac{1}{3!} y_1^3 f^{(3)}(y_0). \tag{13}$$

Equation (8) and equation (9) are substituted into differential equations after applying Laplace transform for each term of differential equations. The Laplace transform and inverse Laplace transform will be used to obtain the value of each term of $y_n$ and $A_n$ [7].

Table of Laplace transforms can be used to avoid the need to apply the definition every time instead of looking up the transform in a table. The transform of the solution of a differential equation in this study are shown as follow [3]:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a, \tag{14}$$

$$\mathcal{L}\left\{\frac{e^{at} - e^{bt}}{a-b}\right\} = \frac{1}{(s-a)(s-b)}, s > a, b. \tag{15}$$

Since there are many functions that have its transform, equation (14) and equation (15) are the basic transforms that are used to solve the problems in this study.

## 2.2 Runge-Kutta Fourth Order (RK4) Method

Consider the first order initial value problem containing of the differential equation [5]

$$\frac{dy}{dt} = f(t, y), \tag{16}$$

and the initial condition [5]

$$y(t_0) = y_0. \tag{17}$$

where $f$ is a known function of two variables, and $t_0$ and $y_0$ are known values. The independent variable $t$ as time and $y$ is the dependent variable. $t_0$ is a given scalar value and known as the initial points while $y_0$ is known as the initial value. The step size $h$ is determined and begins with the initial values $t = t_0$ and $y = y_0$. The four intermediate slopes, $k_1$, $k_2$, $k_3$ and $k_4$ are calculated for $n = 0, 1, 2, 3, \cdots, N-1$, where $N$ is number of steps [5]:

$$k_1 = h\,f(t_n, y_n), \tag{18}$$

$$k_2 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \tag{19}$$

$$k_3 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \tag{20}$$

$$k_4 = h\,f(t_n + h, y_n + k_3). \tag{21}$$

The values of $t_{n+1}$ and $y_{n+1}$ are updated using the weighted average of the slopes [5]:

$$t_{n+1} = t_n + h, \tag{22}$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{23}$$

The value of $y_{n+1}$ is the approximation of $y(t_{n+1})$ and the next value of $y_{n+1}$ is figured out by the current value, which is $y_n$ plus the weighted average of four increments [4].

# 3. Results and Discussion

This section describes the problem of exponential model and logistic model which will be solved by Laplace transform and the RK4 method. For the RK4 method, there are two different step sizes, which are $h = 10$ and $h = 1$. The step size, $h = 10$ represents the population of a town every 10 years and similarly, the step size, $h = 1$ represents the population of a town every year. The absolute errors for each iteration are calculated. Test problem 1 is taken from [12], while test problem 2 is taken from [2].

## 3.1 Test Problem 1

The population of a town grows at a rate proportional to the population present at time $t$. The initial population of 500 increases by 15% in 10 years [12].

**Solution**

Let $N(t)$ is the population of a town at time, $t$ in years. Differential equation can be expressed from equation (1) and it can be solved by Laplace transform and the RK4 method.

The Laplace transform is applied on both sides of equation (1) and use the derivative of Laplace transform, we will get

$$s\mathcal{L}\{N(t)\} - N(0) = k\mathcal{L}\{N(t)\}. \tag{24}$$

$\mathcal{L}\{N(t)\}$ is factorised, substituted with initial population, $N(0) = 500$, and expressed in terms of $s$ and $k$:

$$\mathcal{L}\{N(t)\} = \frac{500}{s-k}. \tag{25}$$

Inverse Laplace transform is applied and refer to equation (14), we will get the solution

$$N(t) = 500e^{kt}. \tag{26}$$

To find the rate constant, $k$, we substitute $N(10) = 575$ to equation (25), we obtain $k = 0.0139762$. Then, we obtain the differential equation

$$\frac{dN(t)}{dt} = 0.0139762N(t). \tag{27}$$

We also obtain

$$N(t) = 500e^{0.0139762t}. \tag{28}$$

This problem which is solved using Laplace transform gives an exact solution.

Equation (27) is taken to solve this problem by RK4 method with the initial population, $N(0) = 500$. The four intermediate slopes, $k_1$, $k_2$, $k_3$ and $k_4$ are formulated:

$$k_1 = h\,[0.0139762N(t)], \tag{29}$$

$$k_2 = h\left[0.0139762\left(N(t) + \frac{k_1}{2}\right)\right], \tag{30}$$

$$k_3 = h\left[0.0139762\left(N(t) + \frac{k_2}{2}\right)\right], \tag{31}$$

$$k_4 = h\,[0.0139762(N(t) + k_3)]. \tag{32}$$

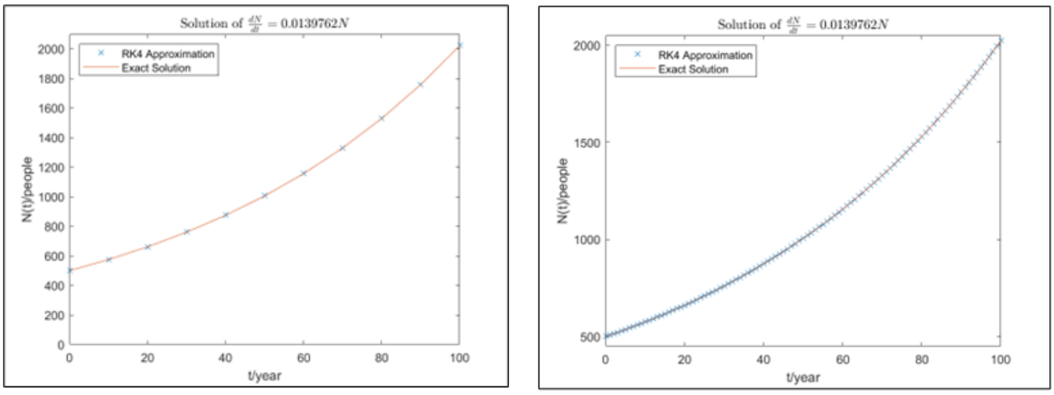
The values of $t_{n+1}$ and $N_{n+1}$ are updated based on the values of $k_1$, $k_2$, $k_3$ and $k_4$ by using equation (22) and equation (23).

**Table 1** *Table of population of a town over time $t$ at $h = 10$ for Test Problem 1*

| $n$ | $t_n$/year | Exact Solution/people (Laplace Transform) [12] | $N_n$/people | Absolute Error of RK4 |
|---|---|---|---|---|
| 0 | 0.0 | 500.000000 | 500.000000 | 0.000000e+00 |
| 1 | 10.0 | 575.000033 | 574.999806 | 2.274759e-04 |
| 2 | 20.0 | 661.250076 | 661.249553 | 5.231945e-04 |
| 3 | 30.0 | 760.437631 | 760.436729 | 9.025104e-04 |
| 4 | 40.0 | 874.503327 | 874.501943 | 1.383849e-03 |
| 5 | 50.0 | 1005.678884 | 1005.676894 | 1.989283e-03 |
| 6 | 60.0 | 1156.530783 | 1156.528037 | 2.745210e-03 |
| 7 | 70.0 | 1330.010477 | 1330.006794 | 3.683156e-03 |
| 8 | 80.0 | 1529.512136 | 1529.507296 | 4.840719e-03 |
| 9 | 90.0 | 1758.939058 | 1758.932796 | 6.262679e-03 |
| 10 | 100.0 | 2022.780033 | 2022.772031 | 8.002311e-03 |

**Table 2** *Table of population of a town over time $t$ at $h = 1$ for Test Problem 1*

| $n$ | $t_n$/year | Exact Solution/people (Laplace Transform) [12] | $N_n$/people | Absolute Error of RK4 |
|---|---|---|---|---|
| 0 | 0.0 | 500.000000 | 500.000000 | 0.000000e+00 |
| 10 | 10.0 | 575.000033 | 575.000033 | 2.525678e-08 |
| 20 | 20.0 | 661.250076 | 661.250076 | 5.809045e-08 |
| 30 | 30.0 | 760.437631 | 760.437631 | 1.002059e-07 |
| 40 | 40.0 | 874.503327 | 874.503326 | 1.536490e-07 |
| 50 | 50.0 | 1005.678884 | 1005.678883 | 2.208701e-07 |
| 60 | 60.0 | 1156.530783 | 1156.530782 | 3.048010e-07 |
| 70 | 70.0 | 1330.010477 | 1330.010476 | 4.089411e-07 |
| 80 | 80.0 | 1529.512136 | 1529.512136 | 5.374654e-07 |
| 90 | 90.0 | 1758.939058 | 1758.939057 | 6.953458e-07 |
| 100 | 100.0 | 2022.780033 | 2022.780033 | 8.884972e-07 |




**(a)** **(b)**

**Fig. 1** *Graph of population of a town over time t for Test Problem 1 (a) for h=10; (b) for h=1*

For the first case $h = 10$, Table 1 and Fig. 1(a) show the population of a town every 10 years. The population in 100 years is 2022.780033 people and we obtain the result by RK4 method is 2022.772031 people with absolute error 8.002311e-03. Fig. 1(a) indicates that the population of a town increases every 10 years and the RK4 method is approximate to exact solution. For the second case $h = 1$, Table 2 and Fig. 1(b) show the population of a town every year. The population in 100 years is 2022.780033 people, which is the same as the result by using RK4 method with absolute error 8.884972e-07. Fig. 1(b) indicates that the population of a town increases every year and RK4 method is approximate to exact solution. In this problem, the solution using Laplace transform is an exact solution, the absolute errors are zero. We can observe that the absolute error for $h = 10$ is more than the absolute error for $h = 1$. It is clear that small step size gives more accurate solutions.

## 3.2 Test Problem 2

The population of the world was about 5.3 billion in 1990. Birth rates in the 1990s ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for the world population is 100 billion [2].

**Solution**

Let $N(t)$ is the population of the world at time $t$ in years. The growth rate of the population, $r$ can be calculated as follow [2]:

$$r = \frac{0.035 - 0.015}{5.3} = \frac{1}{265}. \tag{33}$$

Differential equation can be expressed from equation (3) by substituting equation (33) and carrying capacity for world population, $K = 100$:

$$\frac{dN(t)}{dt} = \frac{1}{265} N(t)\left(1 - \frac{N(t)}{100}\right), \tag{34}$$

where $N(t)$ in billions. Equation (34) is expanded, and we will obtain

$$\frac{dN(t)}{dt} = \frac{N(t)}{265} - \frac{N(t)^2}{26500}. \tag{35}$$

Equation (35) can be solved using Laplace transform and the RK4 method is easier after equation (34) is expanded.

The Laplace transform is applied on both sides of equation (34) and use the derivative of Laplace transform, we will get

$$s\mathcal{L}\{N(t)\} - N(0) = \frac{1}{265}\mathcal{L}\{N(t)\} - \frac{1}{26500}\mathcal{L}\{N(t)^2\}. \tag{36}$$

$\mathcal{L}\{N(t)\}$ is factorized and substituted with initial population, $N(0) = 5.3$, then expressed in terms of $s$ and $\mathcal{L}\{N(t)^2\}$:

$$\mathcal{L}\{N(t)\} = \frac{1404.5}{265s - 1} - \frac{1}{26500s - 100}\mathcal{L}\{N(t)^2\}. \tag{37}$$

By definition of Laplace transform, $N(t)^2$ are not able to solve by Laplace transform. By helping with Adomian polynomials, the nonlinear operator can be decomposed [7]. From equation (8) and equation (9), we have

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} N_n\right\} = \frac{1404.5}{265s-1} - \frac{1}{26500s-100}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\}. \tag{38}$$

The first few polynomials are obtained using equation (10) until equation (13):

$$A_0 = N_0^2, \tag{39}$$
$$A_1 = 2N_0N_1, \tag{40}$$
$$A_2 = 2N_0N_2 + N_1^2, \tag{41}$$
$$A_3 = 2N_0N_3 + 2N_1N_2. \tag{42}$$

If we match both sides of equation (38), we obtain the iterative scheme:

$$\mathcal{L}\{N_0\} = \frac{1404.5}{265s-1}, \tag{43}$$
$$\mathcal{L}\{N_1\} = -\frac{1}{26500s-100}\mathcal{L}\{A_0\}, \tag{44}$$
$$\mathcal{L}\{N_2\} = -\frac{1}{26500s-100}\mathcal{L}\{A_1\}, \tag{45}$$
$$\mathcal{L}\{N_3\} = -\frac{1}{26500s-100}\mathcal{L}\{A_2\}. \tag{46}$$

Inverse Laplace transform is applied and based on equation (14), we will get

$$N_0 = 5.3e^{\frac{t}{265}}. \tag{47}$$

Equation (47) is substituted into equation (39) yields

$$A_0 = 28.09e^{\frac{2t}{265}}. \tag{48}$$

Equation (48) is substituted into equation (44), and based on equation (14) gives

$$\mathcal{L}\{N_1\} = -\frac{28.09}{26500\left(s-\frac{1}{265}\right)}\left(\frac{1}{s-\frac{2}{265}}\right). \tag{49}$$

Inverse Laplace transform is applied, and according to equation (15) produces

$$N_1 = -1.06\times 10^{-3}\left(\frac{e^{\frac{t}{265}} - e^{\frac{2t}{265}}}{\frac{1}{265}-\frac{2}{265}}\right). \tag{50}$$

Equation (50) is simplified to becomes

$$N_1 = 0.2809\left(e^{\frac{t}{265}} - e^{\frac{2t}{265}}\right). \tag{51}$$

Based on equation (45), $A_1$ is obtained first before produce $N_2$:

$$A_1 = 2.97754\left(e^{\frac{2t}{265}} - e^{\frac{3t}{265}}\right). \tag{52}$$

The same procedures to obtain $N_1$ are repeated to yields $N_2$:

$$N_2 = 0.0148877e^{\frac{t}{265}} - 0.0297754e^{\frac{2t}{265}} + 0.0148877e^{\frac{3t}{265}}. \tag{53}$$

Based on equation (46), $A_2$ is obtained first before produce $N_3$:

$$A_2 = 0.23671443e^{\frac{2t}{265}} - 0.47342886e^{\frac{3t}{265}} + 0.23671443e^{\frac{4t}{265}}. \quad (54)$$

The same procedures to obtain $N_1$ are repeated to yields $N_3$:

$$N_3 = 7.890481 \times 10^{-4}e^{\frac{t}{265}} - 2.3671443 \times 10^{-3}e^{\frac{2t}{265}} + 2.3671443 \times 10^{-3}e^{\frac{3t}{265}} - 7.890481 \times 10^{-4}e^{\frac{4t}{265}}. \quad (55)$$

From equation (8), we can define

$$N = N_0 + N_1 + N_2 + N_3. \quad (56)$$

We will yields

$$N(t) = 5.596576748e^{\frac{t}{265}} - 0.3130425443e^{\frac{2t}{265}} + 0.0172548443e^{\frac{3t}{265}} - 7.890481 \times 10^{-4}e^{\frac{4t}{265}}. \quad (57)$$

The exact solution for this problem is

$$N(t) = \frac{\frac{5300}{947}e^{\frac{t}{265}}}{1 + \frac{53}{947}e^{\frac{t}{265}}}. \quad (58)$$

which is different from equation (57) that is solved by Laplace transform. The differences between Laplace transform and exact solution are compared in Table 3 and Table 4.

Equation (35) is taken to solve this problem by RK4 method with the initial population, $N(0) = 5.3$. The four intermediate slopes, $k_1$, $k_2$, $k_3$ and $k_4$ are formulated:

$$k_1 = h\left[\frac{N(t)}{265} - \frac{N(t)^2}{26500}\right], \quad (59)$$

$$k_2 = h\left[\frac{N(t) + \frac{k_1}{2}}{265} - \frac{\left(N(t) + \frac{k_1}{2}\right)^2}{26500}\right], \quad (60)$$

$$k_3 = h\left[\frac{N(t) + \frac{k_2}{2}}{265} - \frac{\left(N(t) + \frac{k_2}{2}\right)^2}{26500}\right], \quad (61)$$

$$k_4 = h\left[\frac{N(t) + k_3}{265} - \frac{(N(t) + k_3)^2}{26500}\right]. \quad (62)$$

The values of $t_{n+1}$ and $N_{n+1}$ are updated based on the values of $k_1$, $k_2$, $k_3$ and $k_4$ by using equation (22) and equation (23).

**Table 3** *Table of population of a town over time $t$ at $h$=10 for Test Problem 2*

| $n$ | $t_n$/year | Exact Solution /billion [2] | Laplace Transform | | RK4 Method | |
|---|---|---|---|---|---|---|
| | | | $N_n$/billion | Absolute Error | $N_n$/billion | Absolute Error |
| 0 | 0.0 | 5.300000 | 5.300000 | 0.000000e+00 | 5.300000 | 0.000000e+00 |
| 1 | 10.0 | 5.492626 | 5.492626 | 1.318504e-09 | 5.492626 | 2.437881e-09 |
| 2 | 20.0 | 5.691833 | 5.691833 | 3.058061e-09 | 5.691833 | 5.016255e-09 |
| 3 | 30.0 | 5.897813 | 5.897813 | 1.112266e-08 | 5.897813 | 7.739136e-09 |
| 4 | 40.0 | 6.110765 | 6.110765 | 3.566817e-08 | 6.110765 | 1.061045e-08 |
| 5 | 50.0 | 6.330888 | 6.330888 | 9.476272e-08 | 6.330888 | 1.363404e-08 |
| 6 | 60.0 | 6.558387 | 6.558387 | 2.178189e-07 | 6.558387 | 1.681362e-08 |
| 7 | 70.0 | 6.793468 | 6.793468 | 4.501450e-07 | 6.793468 | 2.015276e-08 |
| 8 | 80.0 | 7.036341 | 7.036340 | 8.589246e-07 | 7.036341 | 2.365490e-08 |
| 9 | 90.0 | 7.287218 | 7.287216 | 1.541012e-06 | 7.287218 | 2.732327e-08 |
| 10 | 100.0 | 7.546313 | 7.546311 | 2.633027e-06 | 7.546313 | 3.116092e-08 |

**Table 4** *Table of population of a town over time $t$ at $h = 1$ for Test Problem 2*

| $n$ | $t_n$/year | Exact Solution /billion [2] | Laplace Transform | | RK4 Method | |
|---|---|---|---|---|---|---|
| | | | $N_n$/billion | Absolute Error | $N_n$/billion | Absolute Error |
| 0 | 0.0 | 5.300000 | 5.300000 | 0.000000e+00 | 5.300000 | 0.000000e+00 |
| 10 | 10.0 | 5.492626 | 5.492626 | 1.318504e-09 | 5.492626 | 2.513545e-13 |
| 20 | 20.0 | 5.691833 | 5.691833 | 3.058061e-09 | 5.691833 | 5.151435e-13 |
| 30 | 30.0 | 5.897813 | 5.897813 | 1.112266e-08 | 5.897813 | 7.940315e-13 |
| 40 | 40.0 | 6.110765 | 6.110765 | 3.566817e-08 | 6.110765 | 1.089795e-12 |
| 50 | 50.0 | 6.330888 | 6.330888 | 9.476272e-08 | 6.330888 | 1.397993e-12 |
| 60 | 60.0 | 6.558387 | 6.558387 | 2.178189e-07 | 6.558387 | 1.723066e-12 |
| 70 | 70.0 | 6.793468 | 6.793468 | 4.501450e-07 | 6.793468 | 2.065903e-12 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 80 | 80.0 | 7.036341 | 7.036340 | 8.589246e-07 | 7.036341 | 2.425615e-12 |
| 90 | 90.0 | 7.287218 | 7.287216 | 1.541012e-06 | 7.287218 | 2.803979e-12 |
| 100 | 100.0 | 7.546313 | 7.546311 | 2.633027e-06 | 7.546313 | 3.195666e-12 |

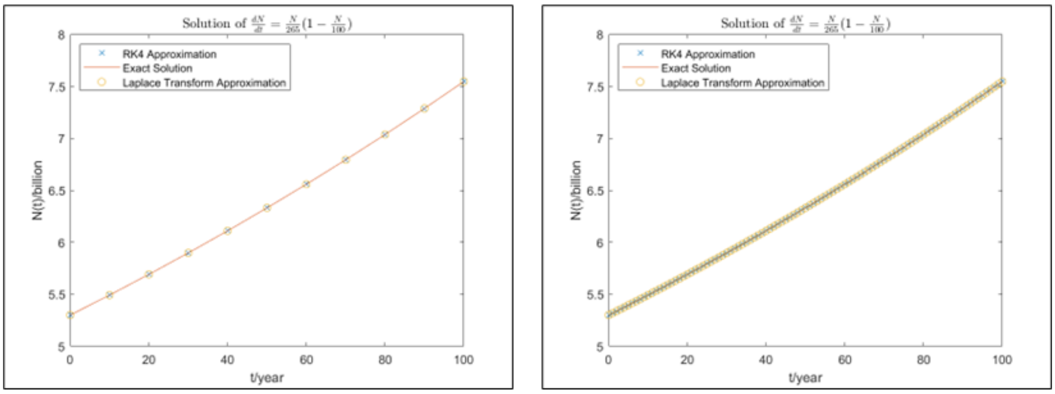



**Fig. 2** *Graph of population of a town over time t for Test Problem 2 (a) for h=10; (b) for h=1*

For the first case $h = 10$, Table 3 and Fig. 2(a) show the population of a town every 10 years. The population in 100 years is 7.546313 billion people, which is same as the result by using RK4 method with absolute error 3.116092e-08 but the result by using Laplace transform is 7.546311 billion people with absolute error 2.633027e-06. Fig. 2(a) indicates that the population of a town increases every 10 years, and Laplace transform and RK4 method are approximate to exact solution. For the second case $h = 1$, Table 4 and Fig. 2(b) show the population of a town every year. The population in 100 years is 7.546313 billion people, which is same as the result by using RK4 method with absolute error 3.195666e-12. The result by Laplace transform is 7.546311 billion people with relative error 2.633027e-06. Fig. 2(b) indicates that the population of a town increases every year, and Laplace transform and RK4 method are approximate to exact solution. We can observe that the absolute error for $h = 10$ is more than the absolute error for $h = 1$. It is clear that small step size gives more accurate solutions.

There are two test problems in this study. Test Problem 1 is an exponential model and Test Problem 2 is a logistic model. These two test problems are solved by using Laplace transform and the RK4 method. The problems are solved based on the properties of Laplace transform. Adomian polynomials are applied to logistic models because it is a nonlinear ODE. There are two step sizes that are considered when the problem is solved using RK4 method. The step size, $h = 10$ represents the population of a town every 10 years while the step size, $h = 1$ represents the population of a town every year. Smaller step size gives more accurate solutions. The results of solving population growth models by using Laplace transform and RK4 method are comparable based on absolute errors.

## 4. Conclusion

This study focuses on how to solve the population models, which are exponential model and logistic model by using Laplace transform and RK4 method. Laplace transform is an analytical method which gives the solution in equation form. The RK4 method is a numerical method which gives the final solution in numeric. Comparison between the results of solving population growth models by using Laplace transform and RK4 method is based on absolute errors. The graphs are plotted to compare the absolute error of Laplace transform and RK4 method for different step sizes. Therefore, Laplace transform and RK4 method can be used to solve the population growth models. For exponential model, Laplace transform is better than RK4 method because the absolute error of Laplace transform is smaller than RK4 method. For logistic model, RK4 method is better than Laplace transform because the absolute error of RK4 method is smaller than Laplace transform. The recommendation for improvement in this study is the results can be compared with other numerical methods.

## Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

## Author Contribution

*The authors confirm contribution to the paper as follows:* ***study conception and design:*** *Goh Chin Yang, Syahirbanun Isa;* ***data collection:*** *Goh Chin Yang;* ***analysis and interpretation of results:*** *Goh Chin Yang, Syahirbanun Isa;* ***draft manuscript preparation:*** *Goh Chin Yang. All authors reviewed the results and approved the final version of the manuscript.*

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