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A Mathematical Model and Analytical Solution for Newtonian Fluid Contained in Elastic Tube with Variable Radius

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Abstract: Throughout the twentieth century, studies on arterial wave mechanics conducted by mathematicians were focused on wave propagation in a prestressed thin-walled elastic tube filled with different types of blood. In this paper, the main objective is to propose a mathematical model for wave propagation in blood flow. Here, the artery represents a prestressed thin-walled elastic tube with a variable radius and the blood is an incompressible Newtonian fluid. First, the dimensional equations of fluid and tube are reduced to non-dimensional equations by introducing non-dimensional quantities. Then, the dimensionless equations of tube and fluid are reduced to nonlinear differential equations in various orders through the reductive perturbation method. These differential equations are solved to obtain the Korteweg-de Vries (KdV) equation with a variable coefficient, and the analytical solution of the KdV equation is determined. As a result, the wavelength of radial displacement becomes narrower but remains unity when the radius of the artery increases. In addition to this, the wave speed decreases until the terminal velocity is achieved. Lastly, before the blood passes through the origin, the wave trajectory decreases along the space and increases slightly at a certain space as the fluid passes through the tube. After that, the wave trajectory continues decreasing. Finally, the wave speed and wave trajectory are influenced by the radius of the artery.

Keywords: Newtonian Fluid, Elastic Tube with Variable Radius, Reductive Perturbation Method, Korteweg-De Vries (KdV) Equation with Variable Coefficient

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1. Introduction

The cardiovascular system, which is known as the circulatory system, is made up of blood, blood vessels and the heart that flows throughout the body. The cardiovascular system pumps blood from the heart to provide oxygen to the lungs. The heart transmits the oxygenated blood through arteries to the rest of the body. The veins carry deoxygenated blood to the heart to start the circulation process [1]. The arteries transport blood away from the heart and bear the highest blood pressure. They are strong, flexible, and durable. Because arteries are elastic, they passively narrow while the heart relaxes between beats, assisting in the maintenance of blood pressure. Capillaries are very thin-walled vessels that serve as a link between arteries and veins. The properties of the walls of capillaries allow oxygen and nutrients from the blood to reach tissues and the waste materials from the tissues to reach the blood. Vein walls are substantially thinner than those of arteries, owing to the reduced pressure in veins [2].

Thomas Young, a physician that devised the wave theory after establishing the light interference principle in 1808 [3]. He also released his famous article in which he evaluated the speed of a pressure wave in an incompressible liquid confined in an elastic tube [4]. Throughout the twentieth century, there has been numerous studies on arterial wave mechanics by mathematicians. Atabek and Lew [5] investigated the wave propagation through a viscous incompressible fluid contained in an initially stressed elastic tube. The fluid is assumed to be Newtonian. The tube is taken to be elastic and isotropic. For this study, the excitation method is implemented and governing equation is the frequency equation.

Tait and Moodie [6] studied the waves propagation and shock formation in nonlinear elastic and viscoelastic fluid contained in the tube. The vessel is considered as an elastic tube and the blood is treated as an incompressible viscous fluid. The governing equations is solved by the method of characteristic in this study. Besides, Demiray [7] investigated nonlinear waves in the fluid with variable viscosity contained in a prestressed thin elastic tube. The artery is treated as a prestressed thin elastic tube and the blood is considered as an incompressible Newtonian fluid with variable viscosity. The mathematical method involved in this study is reductive perturbation method and the governing equation is Korteweg-deVries–Burgers (KdVB) equation. Moreover, Tay and Demiray [6] studied the nonlinear wave propagation in an elastic tube with a stenosis filled with variable viscosity fluid. The blood is treated as a Newtonian fluid with variable viscosity, and the artery is treated as a prestressed thin-walled elastic tube with a stenosis. The forced Korteweg-de Vries–Burgers (FKdVB) equation with variable coefficients as the evolution equation is obtained by using the reductive perturbation method.

In the existing studies, most studies focused on wave propagation in a prestressed thin-walled elastic tube filled with different type of blood. The blood can be considered as an inviscid fluid, Newtonian fluid with constant viscosity as well as Newtonian fluid with variable viscosity. However, the blood pressure is different due to the diameter of the artery when the blood flow from heart to the other parts of body. Hence, the artery is treated to be a thin-walled prestressed elastic tube with variable radius. The reductive perturbation approach is used to explore the propagation of nonlinear waves in the long wave approximation. The first objective in this study is to develop a mathematical formulation to represent the wave propagation in prestressed thin-walled elastic tube with variable radius filled with Newtonian fluid. Moreover, the second objective is to seek the progressive wave solution for the mathematical formulation and the third objective is to analyse the solution on the variation of radial displacement, wave speed and trajectory of wave.

2. Materials and Methods

In this section, some basic equations for wave propagation of blood flow are given.

2.1 Equation of Tube

For this study, the artery is considered as an incompressible, inhomogeneous and prestressed thin-walled elastic tube with variable radius. Figure 1 illustrates the mathematical model for this study, where

R_0 denotes the radius of circulatory cylinder tube, r_0 denotes the deformed radius at the origin of the coordinate system, u^* denotes the radial displacement, H denotes the initial thickness of the tube material and δ denotes the radius of the artery. The equation of motion of the tube in the radial direction [7] is given by,

$$-\frac{\mu}{\lambda_z}\frac{\partial\Sigma}{\partial\lambda_2}+\mu R_0\frac{\partial}{\partial z^*}\left\{\frac{\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)}{\left[1+\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{\frac{1}{2}}}\frac{\partial\Sigma}{\partial\lambda_1}\right\}+\frac{P^*}{H}(r_0-f^*+u^*)=\rho_0\frac{R_0}{\lambda_z}\frac{\partial^2u^*}{\partial t^{*2}} \qquad Eq.1$$

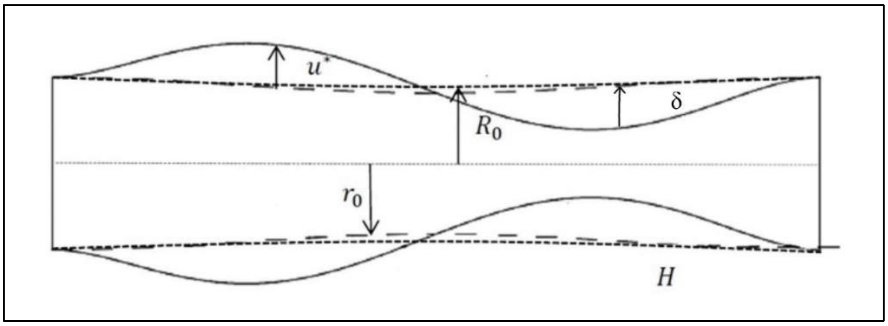


Figure 1: The geometry of the artery with variable radius [8]

where μ is the shear modulus of the tube material, λ_z is the axial stretch of the tube, Σ is the strain energy density function, λ_1 is the stretch ratio along the meridional curve, λ_2 is the stretch ratio along the circumferential curve, z^* is the axial coordinate after static deformation, f^* is the function that characterizes the variation of the radius, P^* is the inner pressure applied by the fluid, ρ_0 is the mass density of the tube and t^* is the time parameter.

2.2 Equation of Fluids

The blood is treated as an incompressible Newtonian fluid for the blood flow in artery. Hence, the averaged equations of motion of an incompressible Newtonian fluid [9] can be expressed as follows,

$$\frac{\partial A^*}{\partial t^*}+\frac{\partial}{\partial z^*}(A^*w^*)=0 \qquad Eq.2$$

$$\frac{\partial w^*}{\partial t^*}+w^*\frac{\partial w^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial P^*}{\partial z^*}-\frac{\bar{\nu}}{\rho_f}\left(-\frac{8w^*}{{r_f}^2}+\frac{\partial^2w^*}{\partial z^{*2}}\right)=0 \qquad Eq.3$$

where A^* is the cross-sectional area of the tube, w^* is the averaged axial fluid velocity, ρ_f is the mass density of the fluid body, P^* is the averaged fluid pressure, $\bar{\nu}$ is the kinematic viscosity of the fluid and $r_f=r_0-f^*+u^*$ is the final radius after deformation. Noting the relation between the cross-sectional area and the final radius, i.e., $A^*=\pi(r_0-f^*+u^*)^2$, then Eq. 2 becomes

$$2\frac{\partial u^*}{\partial t^*}+2w^*\frac{\partial u^*}{\partial z^*}+(r_0-f^*+u^*)\frac{\partial w^*}{\partial z^*}=0. \qquad Eq.4$$

2.3 Nondimensionalizes Equations

Introducing the non-dimensional quantities into the dimensional equations of tube and fluid so that they can be reduced into the non-dimensional equations. Therefore, the following non-dimensional quantities [9] are introduced at this stage,

$$t^*=\left(\frac{R_0}{c_0}\right)t,\qquad z^*=R_0z,\qquad u^*=R_0u,\qquad m=\frac{\rho_0H}{\rho_fR_0},\qquad w^*=c_0w,$$

$$f^* = R_0 f, \qquad r_0 = R_0\lambda_{\theta,} \qquad P^* = \rho_f {c_0}^2 p, \quad {c_0}^2 = \frac{\mu H}{\rho_f R_0}, \qquad \bar{\nu} = \rho_f c_0 R_0 \hat{\nu} \qquad Eq.5$$

where c_0 is a Moens-Korteweg wave speed, and λ_θ is known as initial stretch ratio.

The non-dimensional equations are achieved as follows by introducing Eq. 5 into Eq. 1- Eq. 4 and applying chain rule,

$$p = \frac{1}{\lambda_z \lambda_2}\left\{m\frac{\partial^2 u}{\partial t^2} + \frac{\partial \Sigma}{\partial \lambda_2} - \frac{\partial}{\partial z}\left[\left(-f' + \frac{\partial u}{\partial z}\right)\frac{{\lambda_z}^2}{\lambda_1}\frac{\partial \Sigma}{\partial \lambda_1}\right]\right\} \qquad Eq.6$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} - \hat{\nu}\left(-\frac{8w}{[\lambda_\theta - f + u]^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0 \qquad Eq.7$$

$$2\frac{\partial u}{\partial t} + 2w\frac{\partial u}{\partial z} + (\lambda_\theta - f + u)\frac{\partial w}{\partial z} = 0 \qquad Eq.8$$

2.4 Long Wave Approximation

The propagation of small-but-finite amplitude solitary waves in a prestressed thin-walled elastic tube with variable radius with an incompressible Newtonian fluid is studied. Firstly, apply the reductive perturbation method to obtain various orders of differential equations. Based on Demiray [9], the stretched coordinates are introduced,

$$\xi = \varepsilon^{\frac{1}{2}}(z - gt), \qquad \tau = \varepsilon^{\frac{3}{2}} z, \qquad Eq.9$$

where ε is a small parameter measuring the weakness of nonlinearity and dispersion, g is the scale parameter to be determined from the solution. Assume the function $f(z)$ to be $f(z) = \varepsilon h(\tau)$ and the field variables u, w and p are expressed as asymptotic series in the form as follows,

$$\begin{aligned} u &= \varepsilon u_1(\xi,\tau) + \varepsilon^2 u_2(\xi,\tau) + \cdots \\ w &= \varepsilon w_1(\xi,\tau) + \varepsilon^2 w_2(\xi,\tau) + \cdots \\ p &= p_0 + \varepsilon p_1(\xi,\tau) + \varepsilon^2 p_2(\xi,\tau) + \cdots \end{aligned} \qquad Eq.10$$

Noting the differential relations

$$\frac{\partial}{\partial t} \to -\varepsilon^{\frac{1}{2}} g\frac{\partial}{\partial \xi}\,, \qquad \frac{\partial}{\partial z} \to \varepsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi} + \varepsilon^{\frac{3}{2}}\frac{\partial}{\partial \tau}. \qquad Eq.11$$

By introducing Eq. 10 and Eq. 11 into Eq. 6- Eq. 8 yield various orders of differential equations.

$O(\varepsilon)$ order equations:

$$-g\frac{\partial w_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} = 0 \qquad Eq.12$$

$$-2g\frac{\partial u_1}{\partial \xi} + \lambda_\theta\frac{\partial w_1}{\partial \xi} = 0 \qquad Eq.13$$

$$p_1 = \beta_1[u_1 - h(\tau)] \qquad Eq.14$$

$O(\varepsilon^2)$ order equations:

$$-g\frac{\partial w_2}{\partial \xi} + \frac{\partial p_2}{\partial \xi} + \frac{\partial p_1}{\partial \tau} + w_1\frac{\partial w_1}{\partial \xi} + \nu\frac{8w_1}{{\lambda_\theta}^2} = 0 \qquad Eq.15$$

$$-2g\frac{\partial u_2}{\partial \xi} + \lambda_\theta\frac{\partial w_2}{\partial \xi} + \lambda_\theta\frac{\partial w_1}{\partial \tau} + 2w_1\frac{\partial u_1}{\partial \xi} + u_1\frac{\partial w_1}{\partial \xi} - h(\tau)\frac{\partial w_1}{\partial \xi} = 0 \qquad Eq.16$$

$$p_2 = \left(\frac{mg^2}{\lambda_\theta \lambda_z} - \alpha_0\right)\frac{\partial^2 u_1}{\partial \xi^2} + \beta_1 u_2 + \beta_2 (u_1 - h)^2 \qquad Eq. 17$$

2.5 Solution of Field Equations

The solutions for the various orders of differential equations in Eq. 12- Eq. 17 are sought to obtain the governing equation of the mathematical model. The solutions of Eq. 12- Eq. 14 are given as follows,

$$u_1 = U(\xi, \tau), \qquad w_1 = \frac{2g}{\lambda_\theta}[U + \overline{w}_1(\tau)], \qquad p_1 = \frac{2g^2}{\lambda_\theta}[U - h(\tau)] \qquad Eq. 18$$

where $U(\xi, \tau)$ is an unknown function, whose governing equation will be obtained later and $\overline{w}_1(\tau)$ is the steady flow in the axial direction [9].

Introducing the solutions in Eq. 18 into Eq. 15- Eq. 17 and obtaining the KdV equation with variable coefficient,

$$\frac{\partial U}{\partial \tau} + \mu_1 U \frac{\partial U}{\partial \xi} + \mu_2 \frac{\partial^3 U}{\partial \xi^3} + \mu_3 h(\tau)\frac{\partial U}{\partial \xi} + \mu_4 U = 0, \qquad Eq. 19$$

where the coefficients μ_1, μ_2, μ_3 and μ_4 are

$$\mu_1 = \frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}, \quad \mu_2 = \frac{m}{4\lambda_z} - \frac{\alpha_0}{2\beta_1}, \quad \mu_3 = \frac{3}{2\lambda_\theta} - \frac{\beta_2}{\beta_1}, \quad \mu_4 = \frac{4v}{\lambda_\theta g}. \qquad Eq. 20$$

2.6 Progressive Wave Solution

In this section, the analytical solution for the KdV equation with variable coefficients is studied. Firstly, introduce a progressive wave solution to the perturbed KdV equation with variable coefficients [10] in the following form,

$$U = a(\tau)F(\zeta), \qquad Eq. 21$$

$$\zeta = \alpha(\tau)[\xi - \varphi(\tau)], \qquad Eq. 22$$

where $a(\tau)$, $\alpha(\tau)$ and $\varphi(\tau)$ are unknown functions with the variable τ and to be determined from the solution of the differential equation in Eq. 19. Introducing Eq. 21 and Eq. 22 into Eq. 19 yields

$$\begin{aligned}&[a(\tau)\alpha'(\tau)\xi F'(\zeta) - a(\tau)\alpha'(\tau)\varphi(\tau)F'(\zeta) + a'(\tau)F(\zeta) + \mu_4 a(\tau)F(\zeta)]\\ &\quad + [-a(\tau)\alpha(\tau)\varphi'(\tau)F'(\zeta) + \mu_1 a^2(\tau)\alpha(\tau)F(\zeta)F'(\zeta) + \mu_2 a(\tau)\alpha^3(\tau)F'''(\zeta)\\ &\quad + \mu_3 h(\tau)a(\tau)\alpha(\tau)F'(\zeta)] = 0, \qquad Eq. 23\end{aligned}$$

where the primes denote the differentiation of the corresponding quantities with respect to their arguments. By solving Eq. 23, the final solution of KdV equation can be expressed as follows,

$$U = a_0 \, exp\left(-\frac{4}{3}\mu_4 \tau\right) F(\zeta), \qquad Eq. 24$$

where

$$\zeta = \left(\frac{\mu_1 a_0}{12\mu_2}\right)^{\frac{1}{2}} exp\left(-\frac{2}{3}\mu_4\tau\right)\left\{\xi - \frac{\mu_1 a_0}{4\mu_4}\left[1 - exp\left(-\frac{4}{3}\mu_4\tau\right)\right] - \mu_3\left[\frac{1}{\delta} tanh(\delta\tau)\right]\right\}. \qquad Eq. 25$$

By assuming $\zeta = 0$, the trajectory of wave is denoted as

$$W(\xi, \tau) = \xi - \frac{\mu_1 a_0}{4\mu_4}\left[1 - exp\left(-\frac{4}{3}\mu_4\tau\right)\right] - \mu_3\left[\frac{1}{\delta} tanh(\delta\tau)\right]. \qquad Eq. 26$$

The wave speed is given by

$$v_p = \frac{1}{\frac{\mu_1 a_0}{3}\ exp\left(-\frac{4}{3}\mu_4\tau\right) + \frac{\mu_3}{\delta}[\ sech^2(\delta\tau)]}. \qquad Eq. 27$$

2.7 Numerical Results

The constitutive equation proposed by Demiray [11] for soft biological tissues is given as below.

$$\Sigma = \frac{1}{2\alpha}\left\{exp\left[\alpha\left({\lambda_\theta}^2 + {\lambda_z}^2 + \frac{1}{{\lambda_\theta}^2{\lambda_z}^2} - 3\right) - 1\right]\right\} \qquad Eq. 28$$

where α is a material constant. The explicit expressions of the coefficients α_0, β_1 and β_2 [9] are given as follows,

$$\alpha_0 = \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{{\lambda_\theta}^2{\lambda_z}^3}\right)F,$$

$$\beta_1 = \left[\frac{4}{{\lambda_\theta}^5{\lambda_z}^3} + 2\frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{{\lambda_\theta}^3{\lambda_z}^2}\right)^2\right]F,$$

$$\beta_2 = -\frac{10}{{\lambda_\theta}^6{\lambda_z}^3} + \frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{{\lambda_\theta}^3{\lambda_z}^2}\right)\left(1 + \frac{11}{{\lambda_\theta}^4{\lambda_z}^2}\right) + 2\frac{\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{{\lambda_\theta}^3{\lambda_z}^2}\right)^3 F, \qquad Eq. 29$$

where the function F is defined by

$$F = exp\left[\alpha\left({\lambda_\theta}^2 + {\lambda_z}^2 + \frac{1}{{\lambda_\theta}^2{\lambda_z}^2} - 3\right)\right]. \qquad Eq. 30$$

3. Results and Discussion

The graphical output for Eq. 24, Eq. 27 and Eq. 26 are plotted by using the MATLAB software. The value of material constant, $\alpha = 1.948$ [12] and the constant wave amplitude is denoted by a_0 =1. Moreover, the axial stretch of the tube and the stretch ratio in circumferential directions after static deformation are represented as, λ_z =1.4 and λ_θ =1.6. Besides that, the effect of viscosity is treated as, v =0.05.

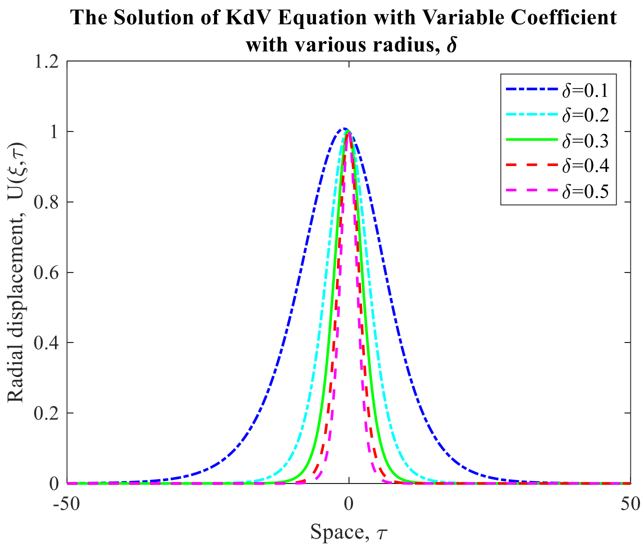


Figure 2: Radial displacement, $U(\xi,\tau)$ versus space, τ with various radius, δ at -50 $\leq \tau \leq$ 50

Figure 2 illustrates the results for the progressive wave solution of KdV equation with variable coefficient for various radius of artery. The solitary wave performed in the bell-shaped form and the amplitude of wave is always unity at the origin as the time, $\xi = 5$. Besides, the width of wave decreases as the value of radius of artery δ increases. However, the amplitude of the bell-shaped solitary wave is remained unity no matter the various values of δ. In general, this phenomenon of the curves is called dispersion, which is the state of getting dispersed or spread for the solitary wave. By comparing the radial displacement with the radius of 0.1 and 0.5, from the figure, the blue dotted line curve with value of radius 0.1 is wider than the purple dotted line curve with value of radius 0.5. This is because the blue dotted line bell-shaped solitary wave (δ=0.1) more dispersive compared to the purple dotted line bell-shaped solitary wave (δ=0.5).

On the other hand, the value of radius of artery δ increases, the width of wave decreases. From the figure, the blue dotted line curve with the value of radius 0.1 has the largest width of wave compared to others. Therefore, it has the lowest frequency of wave.

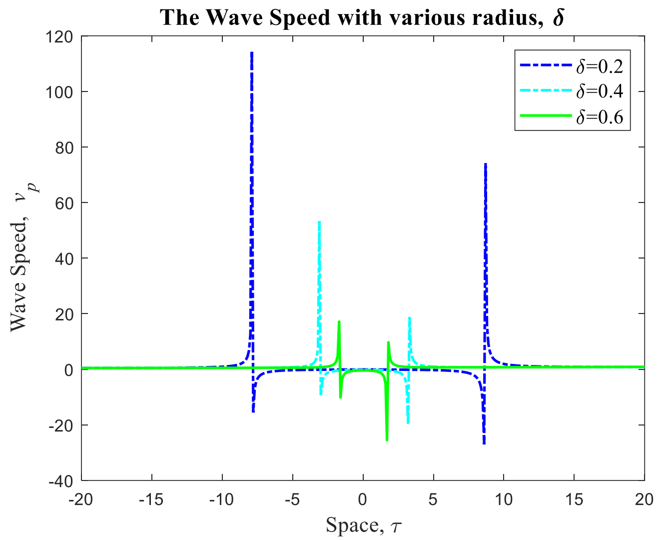
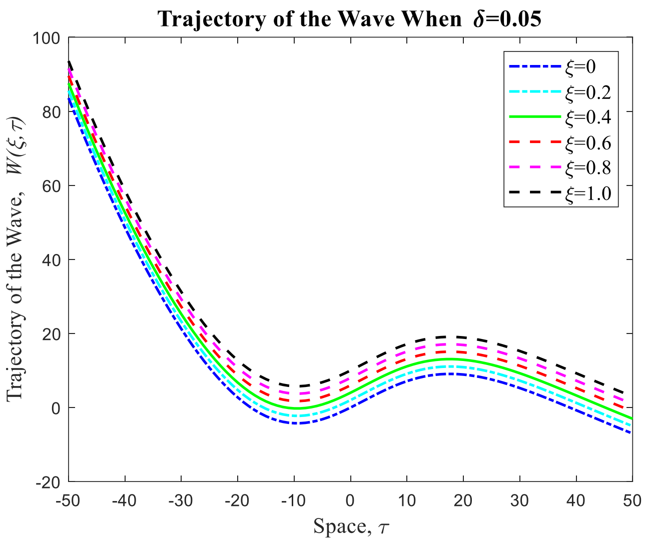


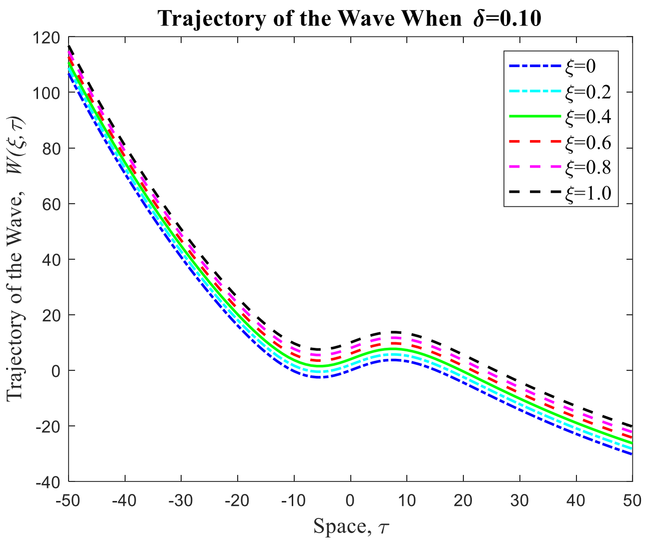
Figure 3: The wave speed, v_p with various radius, δ

Figure 3 represents the output of the wave speed in the artery with various radius. Figure 3 shows that when the value of radius of the artery, δ increases, the wave speed decreases. The lowest value of δ has the highest wave speed compare with the highest value of δ before the fluid passes through the origin. When the blood passes through the origin, the lowest value of δ with highest wave speed travels the longest distance compared with the others.

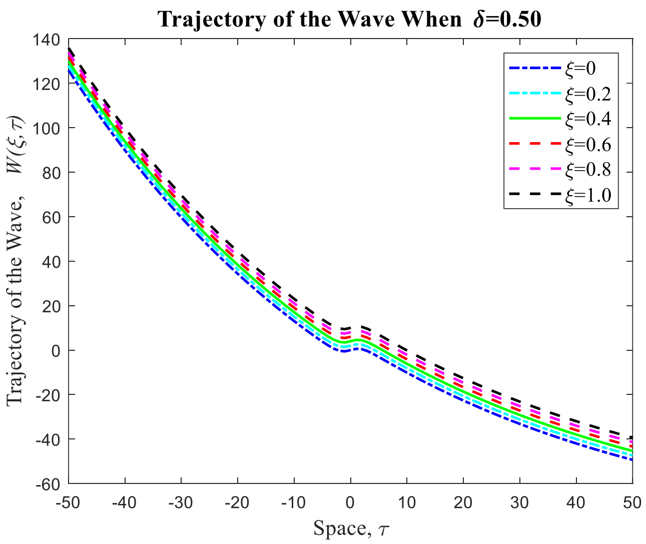
As the fluid passing through the artery with the value of radius is 0.2, it also means that the inner radius of the artery is slightly be compressed due to the outside pressure, the wave speed of fluid is also slightly affected but it is still higher and it able to travel further. However, with the same situation but the value of radius of artery is 0.6, the inner radius of the artery will become narrow the wave speed of the fluid will decrease. This is due to the reason that the wavelength decreases when the inner radius of the artery decrease (also same as the radius of artery, δ increases). Hence, as the wavelength decreases, the wave speed decreases. Lastly, as artery radius increases, wave speed decreases and the wave's ability to travel further is reduced.



(i) Radius, δ =0.05



(ii) Radius, δ =0.1



(iii) Radius, δ =0.5

Figure 4: The trajectory of wave, $W(\xi, \tau)$ with different radius

Figure 4 reveals the trajectory of wave with variable radius of the artery. From the results, the trajectory of wave increases when the time, ξ increases. Besides, it is noticed that when the value of δ increases, the curves of wave trajectory are compressed to each other and it does not make significant difference. Before the blood passes through the origin, the wave trajectory reduces along the space and then slightly increases as the fluid passes through the tube, continuing to decline. This is because of the value of radius of artery, δ increases, the inner radius of the artery become narrower, and the wavelength decreases, thus the frequency of the wave increases.

4. Conclusion

In conclusion, a mathematical model and analytical solution for Newtonian fluid contained in an elastic tube with a variable radius has been studied. The dimensional equations of tube and fluid has been reduced to non-dimensional equations by introducing the dimensionless quantities. The reductive perturbation method has been applied to the dimensionless equations of tube and fluid in order to obtain the various orders of differential equations. The governing equation is the KdV equation with a variable coefficient. From the graphical results, when the radius of the artery increased, the wavelength of radial displacement narrowed but stayed constant. In addition, the wave speed decreased as the artery's radius grew until terminal velocity is reached. Finally, the wave trajectory dropped along the space before the blood passed through the origin, raised slightly at a certain place when the fluid passed through the tube, and then it continued to reduce. Therefore, different radiuses have an impact on wave speed and trajectory.

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