Newton Homotopy Continuation Method for Solving Nonlinear Equations using Mathematica

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Abstract

In this paper, we solve the nonlinear equations by using a classical method and a powerful method. A powerful method known as homotopy continuation method (HCM) is used to solve the problem of classical method. We use Newton-HCM to solve the divergence problem that the classical Newton's method always faces. The divergence problem occurs when a bad initial guess is used. The problem with Newton's method happens when the derivative of given function at initial point equal to zero. The division by zero makes the scheme become nonsense. Thus, an approach used to solve this mathematical problem by using Newton-HCM. The results are implemented by mathematical software known as *Mathematica 7.0*. The results obtained indicate the ability of Newton-HCM to solve this mathematical problem.

Keywords: nonlinear equations; newton homotopy continuation method; homotopy function

1. INTRODUCTION

The root-finding problem is one of the oldest known approximation problems. However the research of this area still continues nowadays. Solving root-finding problems is one important aspect of nonlinear equations. Nonlinear equations consist of several categories such as nonlinear algebraic, trigonometric, exponential, and logarithmic equations. Nonlinear equations can be expressed as

$$f(x)=0 \tag{1}$$

where $x \in \Re$.

There are several methods to solve nonlinear equations such as Newton's method, secant method, bisection method and et cetera. Newton method is also known as Newton-Raphson method. According to Burden and Faires (2011), Newton's method is one of the powerful and well-known numerical methods for solving linear and nonlinear equations. It is already well-known have locally quadratic convergent [1].

There is a major drawback with the Newton's method; hence we need to find the derivative of the function given. This can cause problem if $f'(x)$ is more difficult to calculate than $f(x)$ that already given. The Newton's method does not work when $f'(x_0)=0$ or $f'(x_0)\approx 0$ in which this will lead to the divergence problem or slowly convergence. In this case, homotopy continuation method (HCM) provide a useful approach to find the zeros of each function in term of convergent way in finding the approximate solutions. Homotopy methods will construct a hard or complicated problem into a simpler one with the easier calculation zeros function, $f(x)=0$ [2]

2. METHODOLOGY

Homotopy refers to the homotopy function, $H(x,t)$ and continuous refer to embed the nonlinear equation in a one-parameter family, t, of problems which runs over the interval $t\in[0,1]$ [3].

$$H(x,t)=(1-t)g(x)+tf(x). \tag{2}$$

Since our goal is to solve the $H(x,t)=0$, therefore we have the following conditions:

$$H(x,0)=g(x)=0 \qquad \text{(Initial start system)} \tag{3}$$

$$H(x,1)=f(x)=0 \qquad \text{(Target solution)} \tag{4}$$

There are several ways to identify the auxiliary homotopy function $g(x)$ such as Newton function, fixed-point function and affine function. In this paper, we choose fixed-point function as our auxiliary homotopy function. Mathematically, the fixed-point function is written as

$$g(x)=x-x_0 \tag{5}$$

where x_0 is the initial guess. The basic condition on how to choose auxiliary homotopy function is this function must be controllable and easy to solve.

There are also several HCMs such as Newton-HCM, secant-HCM and Adomian-HCM as described in Wu (2005, 2006, 2007) [4,5,6]. For basic learning purpose, we choose Newton-HCM as our method to solve nonlinear equations. The formula of Newton-HCM can be written as follows

$$x_{i+1} = x_i - \frac{H(x_i,t)}{H'(x_i,t)}, \tag{6}$$

where $i = 0,1,2,...,k-1$ and $t \in [0,1]$. To increase the accuracy of approximate solutions, we use a technique from Palancz et al. (2010)[7].

$$x_{i+1} = \text{NewtonRaphson}(H(x,t_{i+1}),\{x,x_i\}) \tag{7}$$

where x_i is the initial value for calculating next x_{i+1}. However, we are only iterate three times for each t_{i+1}.

ALGORITHM 2.1

To find a solution of $f(x)=0$ by using Newton Homotopy Continuation Method, starting with one initial guess

Input : Function of $f(x)$, Auxiliary function, $g(x)$, Homotopy function, $H(x,t)$
Initial guess, x_0

Output : Approximate solution, $\tilde{x}$.

Step 1 Set t = 0, k = 1 (Initialize Accumulator).

Step 2 Set

$$x_{i+1} = x_i - \frac{H(x_i,t)}{H'(x_i,t)}, \quad f(x,t) = x_{i+1}$$

Step 3 Looping part

For[$w = 1$, Abs[$f[x]$] $\geq 10^{-6}$, $w++$,
$k = k + 1; t = 0; x = a;$
For[$i = 0, i \leq k - 1, i++$,
Print$\left[\text{t=}, t = t + \frac{1.}{k}, \text{,, x}, i + 1, \text{=}, x\right.$
$\left. = \text{SetPrecision}[f[x,t], 15]\right]$
For[$j = 1, j \leq 2, j++$,
Print[" = ", x = SetPrecision[$f[x,t]$,15]]]]];

Step 4 Display result: x_k, Steps
else Break

Step 5 OUTPUT (x_k)
Procedure completed successfully

STOP

3. NUMERICAL EXAMPLE AND DISCUSSION

In this section, we test several examples of scalar of nonlinear equations. The stopping criterion used is $|f(\tilde{x}_k)| < 10^{-6}.$ The equations tested as follows

(i) $f(x) = x^2 + 8x - 9 = 0$ [7] ;

(ii) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 2 = 0$ [5] ;

(iii) $f(x) = x^4 + 6x - 40 = 0$ [8] ;

(iv) $f(x) = x^5 + x^4 - x^2 - 7x + 4 = 0$

(v) $f(x) = x - \cos x = 0$ [1] ;

(vi) $f(x) = \sin x - \frac{\cosh x}{1000} + 0.5 = 0$ [9] ;

(vii) $f(x) = e^{-x} + \cos x = 0$ [10]

The pseudo code of the equation (i) are shown using Mathematica 7.0 as follows;

```
NewtonMethodEq1[x0_] := Module[{a = x0},
  StartTime = SessionTime[];
  Clear[f, x, t];
  f[x_] = x^2 + 8 x - 9;
  K[x_] = x - f[x]/D[f[x], x];
  x = a; t = 0;
  Print["t=0   ,        x0=", x];
  For[i = 1, Abs[f[x]] ≥  10^-6, i++,
   Print["Iteration Number    ", i, ",       x", i, " = ", x = N[K[x], 128]]];
  EndTime = SessionTime[];
  DifferenceTime = EndTime - StartTime;
  Print["Time Used is = ", DifferenceTime, "    second"];
  Print[" x = ", x, " and  f[x] =  ", f[x]]]
```

```
NewtonHomotopyEq1[x0_] := Module[{a = x0},
  StartTime = SessionTime[]; Clear[f, g, H, x, w, t];
  f[x_] = x^2 + 8 x - 9; g[x_] = x - a; H[x_, t_] = (1 - t) (g[x]) + t (f[x]);
  f[x_, t_] = x - H[x, t]/D[H[x, t], x];
  x = a; t = 0; Print["t=0   ,        x0=", x]; k = 1;
  For[w = 1, Abs[f[x]] ≥ 10^-6, w++,
   k = k + 1; t = 0; x = a;
   Print["-----------If number of iterations = ", k, "---------------"];
   For[i = 1, i ≤ k, i++,
    Print["t=", t = t + 1/k, ",       x", i, " = ", x = SetPrecision[f[x, t], 15]];
    For[j = 1, j ≤ 2, j++,
     x = SetPrecision[f[x, t], 15];
     Print["                    ", x];];]];
  EndTime = SessionTime[];
  DifferenceTime = EndTime - StartTime;
  Print["Time Used is = ", DifferenceTime, "    second"];
  Print["Number of Iterations = ", k];
  Print[" x = ", x, " and  f[x] =  ", f[x]]]
```

Then, the output is as follows

```
NewtonMethodEq1[-4]
t=0   ,      x0=-4
Power::infy : Infinite expression 1/0 encountered. >>
Iteration Number   1,     x1 = ComplexInfinity
∞::indet : Indeterminate expression -9 + ComplexInfinity + ComplexInfinity encountered. >>
Time Used is = 0.0040002   second
∞::indet : Indeterminate expression -9 + ComplexInfinity + ComplexInfinity encountered. >>
 x = ComplexInfinity and  f[x] =  Indeterminate
```

There is the divergence problem for Newton's method when started at $x_0 = -4$ (bad initial guess). This divergence problem occurs when the method cannot works at initial stage because there is a division by zero in which $f'(x_0) = 0$. To solve the aforementioned problem, we use Newton-HCM as follows

```
NewtonHomotopyEq1[-4]
t=0  ,      x0=-4
-----------If number of iterations = 2---------------
   1
t=-,     x1 = 21.0000000000000
   2
                 8.74509803921569
                 3.07573184714300
t=1,     x2 = 1.30446763630418
                 1.00873796843647
                 1.00000762188927
-----------If number of iterations = 3---------------
   1
t=-,     x1 = 8.50000000000000
   3
                 2.71296296296296
                 0.541955671157352
   2
t=-,     x2 = 0.761037502224853
   3
                 0.756248389323325
                 0.756246098625721
t=1,     x3 = 1.00624609862517
                 1.00000389650721
                 1.00000000000152
Time Used is = 0.0040003   second
Number of Iterations = 3
 x = 1.00000000000152 and  f[x] =  1.52×10^-11
```

We now take other initial guess $x_0 = -3$ for solving Equation (i). The result is shown in Table 1.

Table 1: Performance of Newton's method and Newton-HCM

Newton's method	Newton-HCM
$x_0 = -3$	$x_0 = -3$
$x_1 = 9$	$x_1 = 0.706347413598374$
$x_2 = 3.4615384615385$	$x_2 = 1.00000000000702$
$x_3 = 1.4060269627280$	
$x_4 = 1.015247601945$	
$x_5 = 1.000023178254$	
$x_6 = 1.000000000054$	

Graphically, it can represented as follows

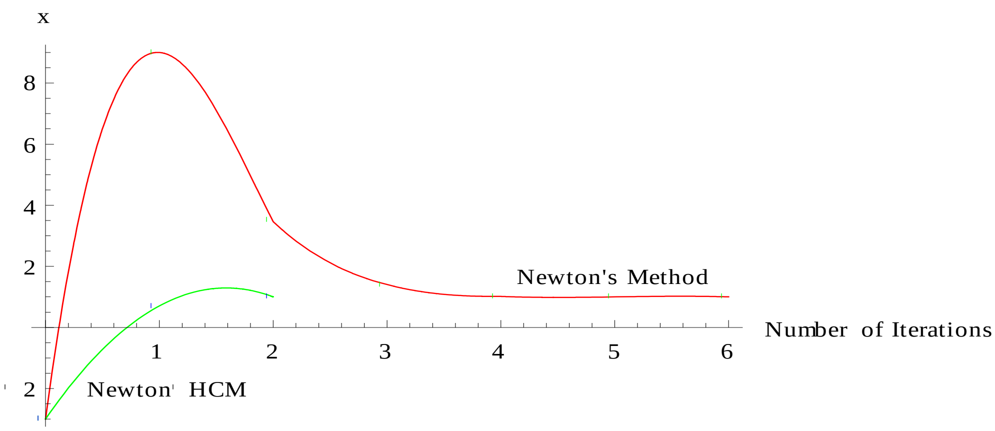

Figure 1: Performance of number of iterations required for Newton's method and Newton-HCM achieve the stopping criterion specified.

The results from Table 1 and Figure 1 show the performance of Newton's method and Newton-HCM when both methods are implemented at $x_0 = -3$. Newton's method requires six iterations whilst Newton-HCM needs only two iterations to converge to the actual root $x = 1$. Other actual root $x = -9$ can be obtained with other initial value as described in Palancz et al. (2010) [7]. From two initial guesses selected, it show the advantage of Newton-HCM over the Newton's method. However, there is not sufficient enough to make a comprehensive conclusion with only one example. Therefore, several equations have been selected and the result shown in Table 2.

Table 2: Comparison between Newton-HCM and Newton's method

Equation	Initial Value x_0	Newton's method	Newton-HCM
(i)	−4	Diverge	3
	−3	6	2
(ii)	−2	Diverge	3
	3	Diverge	3
	3.01	15	3
(iii)	−1	13	4
	0	9	3
(iv)	1	Diverge	3
	−10	12	4
(v)	−2	6	5
	0	4	2
	2	3	2
(vi)	0	3	2
	−4	5	3
(vii)	3	6	3
	0.5	4	3

The same behavior occurs when we do this implementation to the other initial values and equations. Table 2 shows the advantages of the use of Newton-HCM in order to solve nonlinear equations. The advantages are the Newton-HCM can solve the divergence problem at the point of bad initial guess as in Wu (2005) [4]. Furthermore, Newton-HCM has lesser number of iterations rather than classical Newton's method.

4. CONCLUSION

Based on the results for seven examples of nonlinear equations, Newton homotopy continuation method can solve the drawback of Newton's method. Notice that Newton homotopy continuation method can solve the divergence problem for nonlinear equations and it has lesser number of iterations. The number of iterations required is measured based on stopping criterion specified. The best method is measured based on which method has lesser number of iterations. Obviously, we can see this benefit from the result of Newton homotopy continuation method.

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