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Analysis and Simulation of Airfoil with Joukowsky Transformation

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Abstract: The equation of airfoil and its complex potential function can be produced from the Joukowsky transformation. The airfoil will call Joukowsky airfoil because it comes from the transformation. Using Wolfram Mathematica as a software simulator, this transformation produce airfoil with any variant graph and position, also mean camber line and position of trailing edge of airfoil. The complex potential function can also produce equation and graph of equipotential & streamlines.

Keywords: Airfoil, Joukowsky transformation, complex potential function, equipotential, streamlines

1. Introduction

Airfoil, in aerodynamic terms, is a structure with curved surfaces designed to give the most favorable ratio of lift to drag in flight used as the basic form or the wings, fins, and horizontal stabilizer of most aircraft (Fig. 1) [1]. An airfoil-shaped body moving through a fluid produces an aerodynamic force [2]. The component of this force perpendicular to the direction of motion is called lift [3].

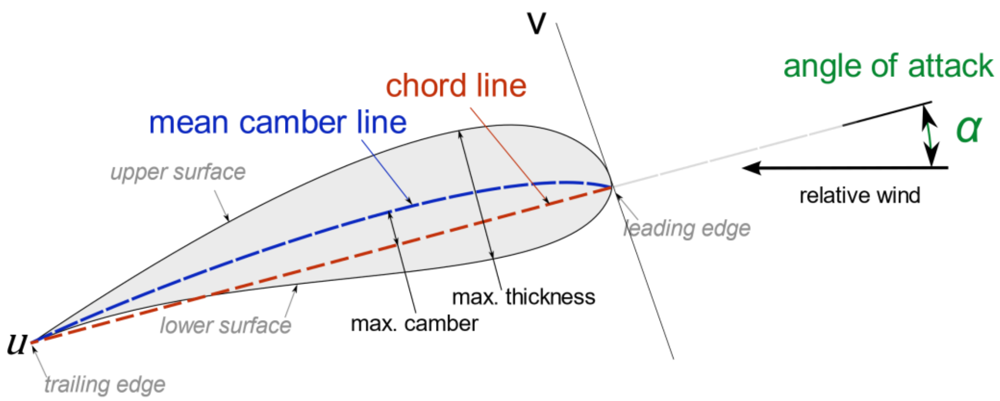

Fig. 1 - Airfoil nomenclature from Anderson [1]

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Joukowsky transformation, in pure and applied mathematics, is a conformal map historically used to understand some principle of airfoil design [1]. There are basic form and other differentiation of this transformation [4], for this paper the transformation is

$$w(z) = z + \frac{b^2}{z}, \tag{1.1}$$

where b as constant with $b \in \mathbb{R}$ and $b > 0$.

The last researches find that the airfoil will come from a circle as a domain [5] and another find that this transformation is as three function composition [6]. Munoz, et al. [7] have shown the element between pure complex analysis and fluid dynamics and present the application in theory of laminar and turbulent flows.

With assumption from [8] with complex analysis in pure mathematics, this paper will explain about airfoil equation and fluid flow around of airfoil with equipotential lines and streamlines line from harmonic function, consist of velocity potential function ϕ and stream function ψ are equals to constant. The complex potential function of fluid flow is in form $f = \phi + i\psi$.

2. Equation of Airfoil

The following theorem shows the equation of airfoil from Joukowsky transformation with the variation called "Joukowsky airfoil" from the perspective of pure mathematics.

Theorem 1

Assume $w = f(z)$ is Joukowsky transformation. If $S \subset \mathbb{C}$ is a circle with center (x_c, y_c), radius $r \neq 0$, and $b = -x_c + \sqrt{r^2 - y_c^2} > 0$, with $x_c, y_c \neq 0$, then $f(S)$ is Joukowsky airfoil asymmetric at u-axis on w-plane.

Proof

If $S \subset \mathbb{C}$ is a circle, then the equation is

$$z = (x_c + iy_c) + re^{i\theta}. \tag{2.1}$$

Substitute the value of z from (2.1) and b to (1.1) to the form $w = u + iv$, then

$$\begin{aligned} w(z) = &\left(r\cos\theta + x_c + \frac{\left(-x_c + \sqrt{r^2 - y_c^2}\right)^2 (r\cos\theta + x_c)}{(r\cos\theta + x_c)^2 + (r\sin\theta + y_c)^2} \right) \\ &+ i\left(r\sin\theta + y_c - \frac{\left(-x_c + \sqrt{r^2 - y_c^2}\right)^2 (r\cos\theta + x_c)}{(r\cos\theta + x_c)^2 + (r\sin\theta + y_c)^2} \right). \end{aligned} \tag{2.2}$$

If $R = \sqrt{(r\cos\theta + x_c)^2 + (r\sin\theta + y_c)^2}$ at (2.2), then

$$u = \left(1 + \frac{\left(-x_c + \sqrt{r^2 - y_c^2}\right)^2}{R^2} \right)(r\cos\theta + x_c) \tag{2.3}$$

and

$$v = \left(1 + \frac{\left(-x_c + \sqrt{r^2 - y_c^2}\right)^2}{R^2} \right)(r\sin\theta + y_c). \tag{2.4}$$

The graph of (2.3) and (2.4) is an airfoil on w-plane. The asymmetrical airfoil corresponding to u-axis is given by $v(-\theta) \neq -v(\theta)$. ■

The asymmetrical airfoil can be also produced with $b = x_c + \sqrt{r^2 - y_c^2}$, which is a reflection from airfoil with $b = -x_c + \sqrt{r^2 - y_c^2}$ coressponding to v-axis on w-plane. Secondly, airfoil with $-y_c$ is a relection from airfoil with y_c coressponding to u-axis on w-plane.

Theorem 2

Assume $w = f(z)$ is Joukowsky transformation. If $S \subset \mathbb{C}$ is a circle with center $(x_c, 0)$, radius $r \neq 0$, and $b = r - x_c > 0$, with $x_c \neq 0$, then $f(S)$ is Joukowsky airfoil symmetric at u-axis on w-plane.

Proof

If $S \subset \mathbb{C}$ is a circle, then the equation of the circle is

$$z = (x_c + i \cdot 0) + re^{i\theta} = x_c + re^{i\theta}. \qquad (2.5)$$

Then substitute the value of z from (2.5) and b to (1.1) to the form $w = u + iv$, then

$$w(z) = \left(r\cos\theta + x_c + \frac{(r - x_c)^2(r\cos\theta + x_c)}{(r\cos\theta + x_c)^2 + (r\sin\theta)^2}\right) + i\left(r\sin\theta - \frac{(r - x_c)^2(r\sin\theta)}{(r\cos\theta + x_c)^2 + (r\sin\theta)^2}\right). \qquad (2.6)$$

If $R = \sqrt{(r\cos\theta + x_c)^2 + (r\sin\theta)^2}$ at (2.6), then

$$u = \left(1 + \frac{(r - x_c)^2}{R^2}\right)(r\cos\theta + x_c)\ \& \qquad (2.7)$$

and

$$v = \left(1 + \frac{(r - x_c)^2}{R^2}\right)(r\sin\theta). \qquad (2.8)$$

The graph of (2.7) & (2.8) is an airfoil on w-plane. The symmetrical airfoil corresponding to u-axis is given by $v(-\theta) = -v(\theta)$. ■

There are points in domain of the circle $S \subset \mathbb{C}$ at Theorem 1 and Theorem 2, will be mapped by Joukowsky transformation to a singular point, because of the value of $b > 0$. It called cups of airfoil graph, in aerodynamic terms called trailing edge. The location is at $(\pm 2b, 0)$ on w-plane from $(\pm b, 0)$ on z-plane.

For symmetrical airfoil from Theorem 2, chord line and mean camber line is coincident at u-axis on w-plane. For asymmetrical airfoil from Theorem 1, mean camber line can be approximated with added a same circle but with $x_c = 0$ on z-plane. Another cases from Theorem 1, for $0 < a < 1$, $0 < h < 1$ and $h \neq 0$, with $b = 1$, $x_c = h$, $y_c = a$, then take $r = \sqrt{(1 - x_c)^2 + y_c^2}$ for approximated mean camber line on the w-plane.

3. Simulation of Fluid Flow Around of Airfoil

To approximate the fluid flow around of airfoil, it will need flow variation theory from [2] and step by step Joukowsky transformation. For a complex potential function of fluid flow on ζ-plane (st-plane) with constant speed $V_0 \in \mathbb{R}$, angle $\alpha = 0$, and direction to positive s-axis,

$$F_1(\zeta) = V_0\zeta. \qquad (3.1)$$

The outside of the circle $S \subset \mathbb{C}$ with center at $(0,0)$ and radius a on w-plane (uv-plane) is transformed by Joukowsky transformation with injective properties where the part with $v > 0$ to $t > 0$ and the part with $v < 0$ to $t < 0$. The codomain is all area on the w-plane. Using flow variation theorem, the complex potential function on w-plane around a circle with center $(0,0)$ and radius a is

$$F_2(w) = V_0\left(w + \frac{a^2}{w}\right). \qquad (3.2)$$

If translation of Joukowsky transformation with vector $z_c = x_c + iy_c$ at domain on w-plane (uv-plane) is

$$z = (w + z_c) + \frac{b^2}{(w + z_c)}, \tag{3.3}$$

the inverse of (3.3) is

$$w = \frac{z}{2} \pm \sqrt{\left(\frac{z}{2}\right)^2 - b^2} - z_c. \tag{3.4}$$

The value of $w \to z$ when $z \to \infty$, then the positive root will be chosen. Based on Theorem 1, the asymmetric airfoil on z-plane (xy-plane) is transformed to be a circle with radius a and $b = x_c + \sqrt{a^2 - y_c^2}$ on w-plane. Then substitute the value of w from (3.4) to (3.2) and using flow variation theorem on z-plane, is

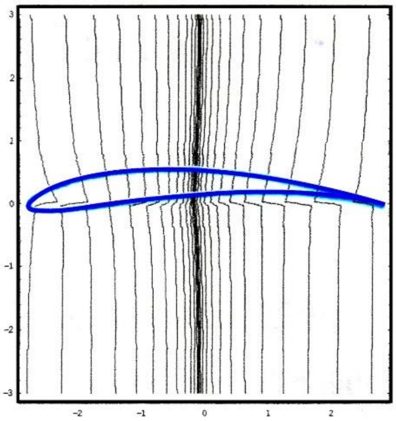
$$F_3(z) = V_0\left(\frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - b^2} - z_c - iy_c + \frac{a^2}{\frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - b^2} - z_c - iy_c}\right). \tag{3.5}$$

The value of $\mathrm{Re}\big(F_3(z)\big) = c = \phi$ and $\mathrm{Im}\big(F_3(z)\big) = k = \psi$, where c and k is constant, is equipotential lines and streamlines lines, respectively. The velocity of fluid $\boldsymbol{V}(x, y)$ arround asymmetrical airfoil is

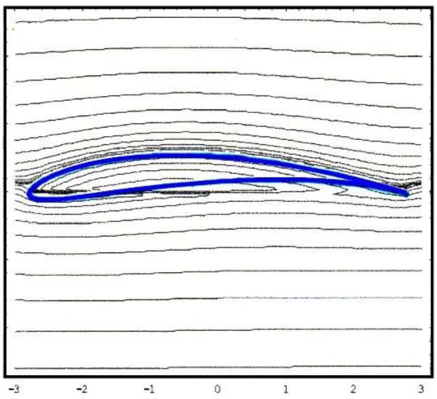
$$\boldsymbol{V}(x,y) = V_0\left(\frac{1}{2} + \frac{\bar{z}}{4\sqrt{-b^2 + \left(\frac{\bar{z}}{2}\right)^2}} - \frac{a^2\left(\frac{1}{2} + \frac{\bar{z}}{4\sqrt{-b^2 + \left(\frac{\bar{z}}{2}\right)^2}}\right)}{\left(-\bar{z}_c + \frac{\bar{z}}{2}\sqrt{-b^2 + \left(\frac{\bar{z}}{2}\right)^2}\right)^2}\right). \tag{3.6}$$

For symmetric airfoil, use the value $y_c = 0$, $b = r + y_c$ and $r = a$.

Using Wolfram Mathematica as a software simulator for value of with $V_0 = 1$, $x_c = -0.1$, $y_c = 0.2$ and $a = r = 1.5$ given an airfoil and its equipotential & streamline (Fig. 2).



(a)



(b)

Fig. 2 - Joukowsky airfoil with (a) equipotential lines and (b) streamlines lines

4. Conclusion

With $w = f(z)$ is Joukowsky transformation, the circle $S \subset \mathbb{C}$ as a domain of the Joukowsky transformation produced Joukowsky airfoil $f(S)$ where is asymmetrical (with $x_c, y_c \neq 0$) or symmetrical (with $x_c \neq 0$, $y_c = 0$) corresponding to u-axis on w-plane. For simulation of the fluid flow around of airfoil, with tools from flow variation theory and Joukowsky transformation given the complex potential function with the velocity of fluid $\boldsymbol{V}(x, y)$, equation of equipotential lines $\phi = \mathrm{Re}\big(F_3(z)\big)$, and streamlines lines $\psi = \mathrm{Im}\big(F_3(z)\big)$. For next research will need deep attention for trailing edge, variation of α, and the force of lift.

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References

[1] Anderson, J. D. (2017). Fundamental of Aerodynamics. Singapore: McGraw-Hill Book Co.

[2] Brown, J. W. and Churchill, R. V. (2014). Complex Variables and Applications Nine Edition. Singapore: McGraw-Hill Book Co. p. 384-389.

[3] Chen, J. and Wang, Q. (2018). Wind Turbine Airfoil and Blades Optimization Design Theory. Beijing: Science Press Ltd.

[4] Levenberg, N. and Wielonsky, F. Zeros of Faber Polynomials for Joukowski Airfoils. (2019). Constructive Approximation (CONSTR APPROX). Available from: Springer Verlag. doi: 10.1007/s00365-019-09482-1

[5] Denker. J. S. (1995). See How It Flies: Perceptions, Procedures, and Principles of Flight. New York: McGraw-Hill Ryerson Limited.

[6] Mathews, J. H. and Howell, R. W. (2012). Complex Analysis for Mathematics and Engineering Sixth Edition. Sudbury: Jones & Bartlett Learning.

[7] Munoz, C. A. A., Alzate, P. P. C. and Granada, J. R. G. (2018). A Survey of Joukowski Airfoil and von Karman Vortex Street. Contemporary Engineering Sciences, Vol. 11, 2018, no. 99, 4921 – 4928, HIKARI Ltd. https://doi.org/10.12988/ces.2018.810540

[8] Spiengel, M. R. (1981). Schaum's Outline of Theory and Problems of Complex Variables SI (Metric) Edition. Singapore: McGraw-Hill Book Co. p. 250-251.