Numerical Computation of Maximum Shear Stress Intensity for a Nearly Circular Crack Subject to Shear Loading

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Abstract

Maximum shear stress intensity for nearly circular cracks subjected to equal and opposite shear stresses are considered. A hypersingular integral equation containing the crack opening displacement is formulated. Conformal mapping technique is employed to transform the obtained hypersingular equation into a similar equation over a circular crack. A suitable collocation points are chosen to reduce the hypersingular integral equation into a system of linear equations. Numerical solution of the linear equations and the corresponding maximum shear intensity is obtained and presented graphically. Our results seem to agree with the existing asymptotic solution.

Keywords: maximum shear intensity; nearly circular crack; conformal mapping; shear load

1. INTRODUCTION

The solution of plane cracks of arbitrary shape inside an isotropic elastic medium has become very interesting and important topic in fracture mechanic. Different approaches have been used by researchers in dealing with the plane crack problems. Mastrojannis et al. [1] formulated the plane crack problem to the solution of a system of two dimensional Fredholm integral equations and numerical solution for tensile mode stress intensity factor is obtained. Singh and Danyluk [2] extended Kassir's work [3,4] in finding the stress intensity factor for coplanar rectangular shaped crack subject to normal loading. Ioakimidis [5] derived a hypersingular integral equation for a flat crack subject to tensile pressure. Qin and Tang [6,7] derived a set of hypersingular integral equation for a flat crack subject to arbitrary loads in three dimensional elasticity using Somigliana formula and finite part integral while a complex hypersingular integral equation was suggested by Linkov and Mogilevskaya [8] to solve the elastic plane problem. Martin [9,10] applied the hypersingular integral equation to solve the crack problems subject to normal and shear loading, respectively and Chen el at. [11] used the hypersingular integral equation to solve the penny shaped crack problem. Theotokoglou [12] applied the hypersingular boundary integral equation method to solve the plane crack in infinite three dimensional bodies under shear loading.

Other approaches are perturbation method by Rice and Gao [13,14], variant formula by Borodachev [15,16] and Frechet derivative of some nonlinear operation [15]. In this study, the hypersingular integral equation approach is used to compute the maximum shear stress intensity for a nerly circular crack and we compare our computational result with Gao's [17].

2. PROBLEM FORMULATION

Consider a three dimensional infinite, homogenous, elastic and isotropic solid body containing a flat circular crack, Ω located on the Cartesian coordinate (x, y, z) with origin O and Ω lies in the plane $z = 0$. Let the radious of the crack, Ω be α and $\Omega = \{(r,\theta) : 0 \le r \le \alpha, -\pi \le 0 < \pi\}$. Assume that the stress at infinity and the body force are absent. Now, equal and opposite shear stresses in the X directions, $q(x, y)$ are applied to the crack plane and on the planes, the shear stress component is given by

$$\tau_{xz} = \frac{\mu}{1-\nu} q_x(x, y)$$

where ν is the Possion's ratio and μ is the shear modulus. Adopted the Sogmigliana formula [18], followed by integration by part and making use of the relationship of Cauchy principle value and hypersingular equation [20], then, the plane crack problem subject to shear loading is given as [10]

$$\frac{1}{8\pi} H.S.\int_{\Pi} \frac{(2-v)+3ve^{j2\Theta}}{R^3} w(x,y) d\Omega = q(x_o, y_o)$$

(1) where Ω denotes the crack domain with boundary $\partial\Omega$, $w(x,y)$ is the unknown crack opening displacement, the over bar denote complex conjugation with respect to $j-$complex, with $j=\sqrt{-1}$, R is defined as

$$R^2 = (x-x_o)^2 + (y-y_o)^2$$

and the angle Θ is

$$x - x_o = R\cos\Theta \text{ and } y - y_o = R\sin\Theta$$

Equation (1) is solved subjected to $w(x,y)=0$ on $\partial\Omega$. The notation of $H.S$ in front of the integral sign on the left hand side of equation (1) should be interpreted as Hadamard finite part integral [21].

3. NEARLY CIRCULAR CRACK

Assume that Ω is an arbitrary shaped crack with smooth boundary with respect to the origin O, such that Ω is written as

$$\Omega = \{(r,\theta): 0 \le r < \rho(\theta) < -\pi\}$$

where the boundary of Ω, $\partial\Omega$ is given by $r = \rho(\theta)$. Next, let the polar coordinate defined as $\zeta = se^{i\varphi}$ where $|\zeta| < 1$, hence, the unit disc D is given by

$$D = \{(s,\varphi): 0 \le s < 1, -\pi \le \varphi < \pi\}$$

Employed the properties of Rieman mapping theorem, Ω is mapped onto the unit disc, D using $z = af(\zeta)$ where

$$f(\zeta) = \zeta + c\zeta^2 \tag{3}$$

The domain is circular if $c = 0$ and as $2|c| \to 1$, a cusp developed.

Substitution equation (3) into (2) gives [16] :

$$\frac{2-v+3ve^{2j\theta}}{8\pi} H.S\int_D \frac{W(\xi,\eta)}{S^3} d\xi d\eta + \frac{2-v}{8\pi} C.P\int_D W(\xi,\eta) K^{(1)}(\zeta,\zeta_o) d\xi d\eta$$
$$+ \frac{3v}{8}\int_D W(\xi,\eta) K^{(2)}(\zeta,\zeta_o) d\xi d\eta = Q(\xi_o,\eta_o) \tag{4}$$

where $S = |\zeta - \zeta_0|$, $K^{(1)}(\zeta,\zeta_o)$ and $K^{(2)}(\zeta,\zeta_o)$ are Cauchy-type and weakly singular kernel [18]

$$K^{(1)}(\zeta,\zeta_o)=\frac{\left|f^{'}(\zeta)\right|^{-\frac{3}{2}}\left|f^{'}(\zeta_o)\right|^{-\frac{3}{2}}}{\left|f(\zeta)-f(\zeta_o)\right|^{3}}e^{j(\delta-\delta_o)}-\frac{1}{\left|\zeta-\zeta_o\right|^{3}}$$

$$K^{(2)}(\zeta,\zeta_o)=\frac{\left|f^{'}(\zeta)\right|^{-\frac{3}{2}}\left|f^{'}(\zeta_o)\right|^{-\frac{3}{2}}}{\left|f(\zeta)-f(\zeta_o)\right|^{3}}e^{j(2\Theta-\delta-\delta_o)}-\frac{1}{\left|\zeta-\zeta_o\right|^{3}}e^{2j\phi}$$

ϕ and δ are defined as $\zeta-\zeta_o=Se^{i\varphi}$ and $f^{'}(\zeta)=\left|f^{'}(\zeta)\right|e^{i\delta}$, respectively. The notation C. P. is denoted as Cauchy Princinple value integral. The hypersingular integral equation over a circular disc D is to be solved subject to $W=0$ on $s=1$.
Now, write W as

$$W(\xi,\eta)=\sum_{n,k}W_k^nY_k^n(s,\varphi) \tag{5}$$

where

$$Y_k^n(s,\varphi)=s^{|n|}C_{2k+1}^{|n|+\frac{1}{2}}(\sqrt{1-s^2})e^{jn\varphi} \tag{6}$$

and

$$\sum_{n,k}=\sum_{n=-N_1}^{N_1}\sum_{k=0}^{N_2}$$

Introduce

$$L_h^m(s,\varphi)=s^{|m|}C_{2h+1}^{|m|+\frac{1}{2}}(\sqrt{1-s^2})\cos(m\varphi) \tag{7}$$

such that the relationship of these two function is

$$\int_{\Omega}\frac{Y_k^n(\zeta)L_h^m(\zeta)sdsd\varphi}{\sqrt{1-s^2}}=B_k^n\delta_{hk}\delta nm \tag{8}$$

where δ_{ij} is the Kroneker delta and

$$B_k^n=\frac{\sigma_n}{2}h_{2k+1}^{n+\frac{1}{2}} \quad ,$$

$$h_{2k+1}^{n+\frac{1}{2}}=\frac{\pi}{2^{2n}}\frac{\Gamma(2k+2n+2)}{(2k+n+\frac{3}{2})(2k+1)!\left[\Gamma(n+\frac{1}{2})\right]^2}$$

$$\sigma_n=\begin{cases}2\pi & n=0\\ \pi & n\neq 0\end{cases}$$

Both functions $L_h^m(s,\varphi)$ and $Y_k^m(s,\varphi)$ have square-root zeros at $s=1$.
Define

$$W_k^n = -\overline{W}_k^n G_{2k+1}^{|n|+\frac{1}{2}} \sqrt{\frac{E_k^n}{B_k^n}}$$

Subsitute equation (5) into (4) and multiply with equation (6) and perform integrateation over D using the relationship of (8) leads to

$$\sum_{n,k} \overline{W}_k^n \left(-\frac{2-v+3ve^{2k\Theta}}{2}\delta_{hk}\delta_{|n||m|} + S_{hk}^{mm}\right) = Q_k^n;$$
$$-N_1 \le m \le N_1; 0 \le k \le N_2 \qquad (9)$$

where

$$S_{hk}^{mm} = \frac{2-v}{8\pi\sqrt{E_k^m B_k^m}\sqrt{E_h^n B_h^n}} T_{hk}^{mm},$$

$$T_{hk}^{mm} = \int_D L_h^m(\zeta) \int_D Y_k^n(\zeta) H(\zeta,\zeta_o) d\zeta d\zeta_o,$$

$$Q_k^n = \frac{1}{\sqrt{E_k^n B_k^n}} \int_D Y_k^n(\zeta_o) Q(\zeta_o) d\zeta_o$$

and

$$H(\zeta,\zeta_o) = (2-v)K^{(1)}(\zeta,\zeta_o) + 3vk^{(2)}(\zeta,\zeta_o).$$

In (9) , we have used the following notation :

$\zeta_o = \zeta_o(s_o,\varphi_o), Q(\zeta_o) = Q(s_o\cos\varphi_o, s_o\sin(\varphi_o))$ and $d\zeta_o = s_o ds d\varphi$

Equation (9) is a system of linear equations and is to be solved for the coefficients, $\overline{W}_k^n$, which will be used later in finding the maximum shear stress intensity. The integration in (9) are all regular, solving them should give no much difficulties. Here, we have used the Gaussian quadrature and trapezoidal formulas for the radial and angular direction, with the appropriate choice of collocation points (s,φ) and (s_o,φ_o) , respectively.

4. MAXIMUM SHEAR STRESS INTENSITY FACTOR

The maximum shear stress intensity factor is defined as [17]

$$K(\varphi) = \sqrt{[K_2(\varphi)]^2 + [K_3(\varphi)]^2} \tag{10}$$

where $K_2(\varphi)$ and $K_3(\varphi)$ are shear and tearing modes stress intensity factor, respectively. The stress intensity factor is defines as

$$K_j(\varphi) = T_j \left\{ \left| f^{'}(e^{i\varphi}) \right|^{-1} \sum_{n,k} \frac{\overline{W}_k^n}{\sqrt{E_k^n B_k^n}} Y_k^n(\varphi) \right\}; j = 2,3 \tag{11}$$

where T_j is constant,

$$Y_k^n(\varphi) = D_{2k+1}^{|n|+\frac{1}{2}}(0)\cos(n\varphi) \text{ and } C_{2k+1}^{|n|+\frac{1}{2}}(\sqrt{1-s^2}) = \sqrt{1-s^2}\, D_{2k+1}^{|n|+\frac{1}{2}}(\sqrt{1-s^2})$$

5. NUMERICAL RESULT

Numerical calculation have been carried out and shows that our numerical scheme converges rapidly with only a small value of N.

Table 1: Numerical convergence maximum stress intensity factor $f(\zeta) = \zeta + 0.1\xi^2$

N	$K(0.00)$	$K_2(\frac{\pi}{4})$	$K(\frac{\pi}{2})$	$K(\frac{3\pi}{4})$	$K(\pi)$
0	1.1397E-06	9.7723E-07	8.1972E-07	1.0053E-06	1.1862E-06
1	0.0000E+0000	0.0000E+0000	1.4571E-06	0.0000E+0000	0.0000E+0000
2	3.0662E-07	5.0149E-26	1.8584E-06	4.6429E-25	3.1914E-07
3	1.2477	1.0551	0.8555	1.0137	1.1790
4	1.2477	1.0551	0.8555	1.0137	1.1790
5		1.0551	0.8555	1.0137	
6			0.8555		

Figure 1, 2 and 3 shows that the variations of $K(\varphi)$ against φ for $c = 0.001$, $c = 0.01$ and $c = 0.1$, respectively. It can be seen that the maximum shear stress intensity has local extremal value when the crack front is at $\cos(\varphi) = \pm 1$ or $\sin(\varphi) = \pm 1$.

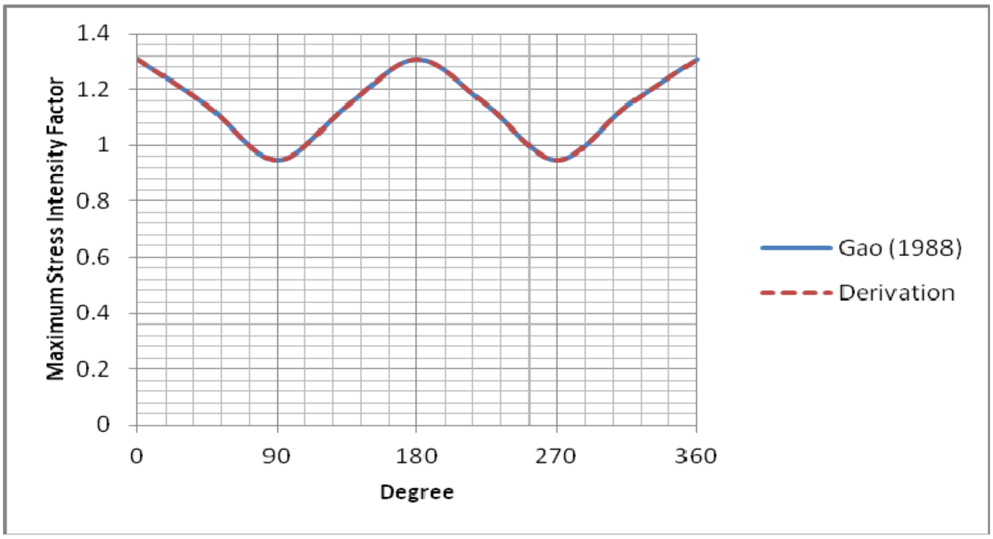

Figure 1: The maximum shear stress intensity for $f(\zeta) = \zeta + 0.001\zeta^2$.

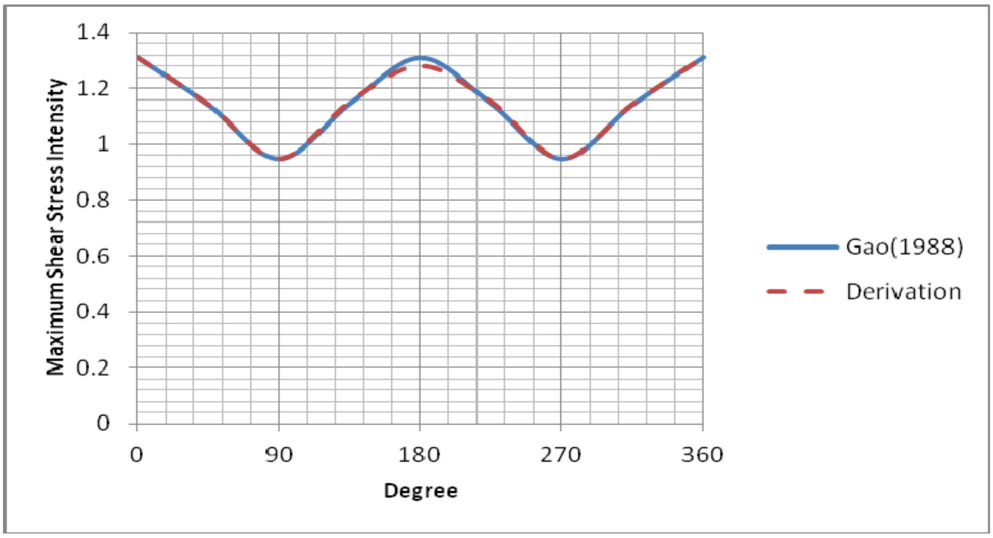

Figure 2: The maximum shear stress intensity for $f(\zeta) = \zeta + 0.01\zeta^2$.

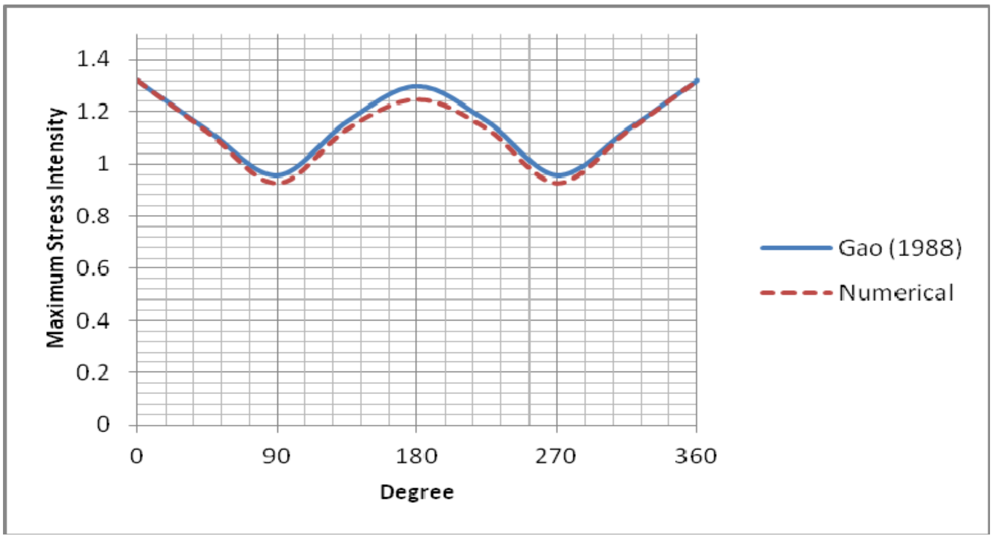

Figure 3: The maximum shear stress intensity for $f(\zeta)=\zeta+0.1\zeta^{2}$.

6. CONCLUSION

In this study, a hypersingular integral equation for crack problem subject to shear loading is formulated. Then, a nearly circular crack is mapped conformally onto a unit circle where the equation is transformed into a similar hypersingular integral equation over a circular crack, which enable us make use of the formula obtained by Krenk [22]. By choosing the appropriate collocation points, this equation is reduced into a system of linear equations and solved for the unknown coefficients, which are later used in finding the maximum shear stress intensity. The maximum shear stress intensity for the mentioned crack subject to shear load presented graphically. Our result seems to agree with the previous work by Gao [17].

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