Newton Homotopy Solution for Nonlinear Equations Using Maple14

Nor Hanim Abd. Rahman, Arsmah Ibrahim, Mohd Idris Jayes

Faculty of Computer and Mathematical Sciences,
Universiti Teknologi MARA Malaysia (UiTM),
40450 Shah Alam, Selangor, Malaysia

*Corresponding email: norhanim@ppinang.uitm.edu.my

Abstract

Many numerical approaches have been suggested to solve nonlinear problems. Some of the methods utilize successive approximation procedure to ensure every step of computing will converge to the desired root and one of the most common problems is the improper initial values for the iterative methods. This study evaluates Palancz et.al's. (2010) paper on solving nonlinear equations using linear homotopy method in Mathematica. In this paper, the Newton-homotopy method using start-system is implemented in Maple14, to solve several nonlinear problems. Comparisons of results obtained in terms of number of iterations and convergence rates show promising application of the Newton-homotopy method for nonlinear problems.

Keywords: Homotopy, Newton-Raphson, Start-system, Polynomials, *Maple14*.

1. INTRODUCTION

Newton-Raphson method offers fast, accurate numerical results compared to other methods and the best method proven to converge quadratically and applicable to many fields of knowledge. Many attempts have been done in finding the solution for the roots of equations of the form of

$$f(x) = 0; \qquad x \in \Re; \tag{1}$$

where $f(x)$ may be given explicitly as polynomials (Nor Hanim, Arsmah & Mohd Idris, 2010a, 2010b, 2010c; Li, Cheng & Neta, 2010; Saeed & Khthr, 2010; Kou, Wang & Li, 2010; Shenggou, Xiangke & Lizhi, 2009; Yun, 2009; Chun & Neta, 2009; Chun, Bae & Neta, 2009; Fang & He, 2007,2009) or, ordinary or partial differential equations at specific values (He, 1999). However, one of the disadvantages of the classical iterative Newton-Raphson method such as,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{where,} \quad f'(x_i) \neq 0 \tag{2}$$

is the usage of division in the formula which is not very efficient in terms of computational time (Nor Hanim, Arsmah & Mohd Idris, 2010a,2010b,2010c) that could lead to divergent, when $f'(x_i) = 0$ and the usage of derivative is mostly difficult to solve.

Another known method in solving nonlinear problems is homotopy method, which is a global numerical method used in various science and engineering areas (Palancz et.al, 2010; Hazaveh et.al, 2003). Much attention too has been given to develop several iterative methods for solving nonlinear equations such as the Newton homotopy continuation method (Saeed & Khthr, 2010; Rafiq & Awais, 2008; Abbasbandy, 2003). Homotopy can guarantee to converge by certain path if a suitable auxiliary homotopy function is chosen (Rafiq & Awais, 2008).

In this paper, the general algorithms of homotopy equations via Newton-Raphson method are used to find roots of polynomials. The efficiency of these methods is illustrated and comparison of convergence and number of iterations are made.

2. PROJECT DESIGNS

The power form is the standard way for a polynomial in mathematical discussions and very convenient form for differentiating and integrating. However, it may lead to loss of significance. So to avoid this, the shifted power is introduced. Below is the basic theorem of a polynomial.

Definition 2.1:

A polynomial $p(x)$ of degree n is defined as

$$p(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots + a_ix^n \tag{3}$$

where, a_i are real constants for $i = 0$ to n.

2.1 The basic idea of linear homotopy

We consider the following non-linear algebraic equation, $f(x) = 0;$ and define the convex homotopy for the function $H(x,\lambda): \Re \times [0,1] \to \Re$,

$$H(x,\lambda) = (1-\lambda)p(x) + \lambda q(x) = 0; \tag{4}$$

where,

λ is an embedded parameter and $\lambda \in [0,1]$;
$p(x)$ is the start system;
$q(x) = f(x)$ is the target system;
$H(x,0) = p(x)$ & $H(x,1) = q(x) = f(x)$.

There are 3 basic ways to identify a start system of a linear homotopy (Palancz et.al, 2010) such as,

(i) The fixed-point homotopy:

$$p(x) = x - x_0 \ , \qquad H(x,\lambda) = (1-\lambda)(x - x_0) + \lambda q(x) = 0; \tag{5}$$

where x_0 is an initial approximation of Eq.(4).

(ii) The Newton-homotopy:

$$p(x) = q(x) - q(x_0)\ , \qquad H(x,\lambda) = (1-\lambda)\left[q(x) - q(x_0)\right] + \lambda q(x) = 0; \tag{6}$$

(iii) The start-system Newton-homotopy:

$$p(x) = x^n - \mathrm{C}, \qquad H(x,\lambda) = (1-\lambda)(x^n - \mathrm{C}) + \lambda q(x) = 0; \tag{7}$$

where n preferably be the highest power of x of Eq.(3),
C is any constant, and the roots of $p(x) = 0$ can be easily found.

Besides the ease of Newton-homotopy, it does not guarantee to converge (Palancz et.al, 2010; Choi & Book, 1991). Hereafter, our discussions will only proceed with the type-(iii) method for its flexibility to choose the values of n and C. Below are some of the examples of *Maple14* algorithms used. The iterations will follow the following algorithms.

Algorithm 2.1: Newton-homotopy using start-system

Step1: Identify $q(x) = f(x) = 0$.

Step2: Identify $p(x)$, such that $p(x) = x^n - \mathrm{C}$.

Step3: Find the initial value, x_0, by setting $p(x) = x^n - \mathrm{C} = 0$ such as,

- ➢ restart; $q := x \rightarrow x^2 - (1-x)^5$; p := $x \rightarrow x^5 - 1$; *fsolve*$(q(x))$;
- ➢ N := *fsolve*$(p(x))$;

Step4: Simplify $H(x,\lambda) = (1-\lambda)(x^n - \mathrm{C}) + \lambda q(x)$ such as,

- ➢ $H := x \rightarrow (1-\lambda)p(x) + \lambda\, q(x)$; *simplify*$(H(x,\lambda))$;

Step5: Iterate $H(x,\lambda) = (1-\lambda)(x^n - \mathrm{C}) + \lambda q(x)$ where $\lambda \in [0,1]$ e.g. 0.0, 0.2, 0.4, 0.6, 0.8, and 1.0 by using the classical Newton Raphson (See Eq. (2)),

- ➢ DH := D(H); *simplify*$(DH(x,\lambda))$;
- ➢ *newt* := $x \rightarrow$ *evalf*$\left(x - \dfrac{H(x)}{DH(x)}\right)$;

3. NUMERICAL EXAMPLES AND DISCUSSIONS

In this section, some numerical examples presented to illustrate the efficiency of the iterative methods such as the classical Newton-Raphson (CNR) and the start-system homotopy (see Eq. (7)). By using the following test functions ((a), (b), (c), (d) and (e)) and specified the outputs to 10-digit values, we obtained the results as shown in Table 1.

(a) $f_1(x) = x^2 + 8x - 9$ (See Palancz et.al, 2010);

(b) $f_2(x) = x^4 + 6x - 40$;

(c) $f_3(x) = 5x^6 + 3x^4 - x^2 - 12$;

(d) $f_4(x) = x^2 - 3x + 2 - e^x$ (See Abbasbandy, 2003; Saeed & Khthr, 2010);

(e) $f_5(x) = \cos(x) - x$ (See Abbasbandy, 2003; C.Chun et.al, 2009; C.Chun & Neta, 2009; Kou et.al, 2010; Li et.al, 2010).

Table1 shows that the efficiency of the iterative Newton-homotopy using start-system (NHss) gives equal or better results in terms of convergence rate as compared to the classical Newton-Raphson. It seems that the computations converge in less than 5 iterations.

We also approximated using different powers of x on the start-systems and recorded the number of iterations. The following test functions (f) and (g) are used and the approximated zeros are displayed in Table 2:

(f) $f_6(x) = x^2 - (1-x)^5$ (See Javidi, 2009);

(g) $f_7(x) = x^3 + 4x^2 - 10$ (See Saeed & Khthr, 2010).

Table 1: Numerical experiment results of the existing method, the Classical Newton-Raphson (CNR), and the proposed methods, the Newton-Homotopy using start-system (NHss) using *Maple14.*

Functions Used $f(x)=q(x)=0$ **='target system'; &** $H(x,\lambda)=(1-\lambda)p(x)+\lambda q(x)$	**Roots (CNR)** $x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$ **iteration,** i	**Root(s) using Newton-Homotopy Method ;** $p(x)=0$ **= 'start-system'** $\lambda \to H(x,\lambda)$ **(NHss)**	$p(x)=0$ **& Initial guess,** x_0	**Iteration using NHss,** i
$f_1(x)=x^2+8x-9$	-9, 1	$0.0 \to 3 = x_0$	x^2-9	1
&	(4),(4)	$0.2 \to 2.304834939$	&	4
$H_1(x,\lambda)=x^2-9+8\lambda x$		$0.4 \to 1.800000000$	3,-3	4
(as in Palancz et.al, 2010)		$0.6 \to 1.441874542$		4
		$0.8 \to 1.86342440$		4
		$1.0 \to 1.00000000$		4
$f_2(x)=x^4+6x-40$	2.66728462,	$0.0 \to 2.514866859$	x^4-40	1
&	-2.66728462	$0.2 \to 2.466985645$	&	3
$H_2(x,\lambda)=x^4-40+6\lambda x$	(4),(4)	$0.4 \to 2.418214028$	2.514866859	3
		$0.6 \to 2.368563982$	-	4
		$0.8 \to 2.318057459$	2.514866859	4
		$1.0 \to 2.6672846272$		4
$f_3(x)=5x^6+3x^4-x^2-12$		$0.0 \to 1.157093730$	$5x^6-12$	1
&	1.099353175,	$0.2 \to 1.144421529$	&	3
$H_3(x,\lambda)=$	-1.099353175	$0.4 \to 1.132344572$	1.157093730	3
$5x^6-12+13\lambda x^4-\lambda x^2$	(4),(4)	$0.6 \to 1.12089010$	-	4
		$0.8 \to 1.109842258$	1.157093730	4
		$1.0 \to 1.099353175$		4
$f_4(x)=x^2-3x+2-e^x$		$0.0 \to 2.000000000$	x^2-3x+2	1
&	0.2575302854	$0.2 \to 0.6936495745$	&	4
$H_4(x,\lambda)=$	(4)	$0.4 \to 0.5343287354$	1.000000000	*4
$x^2+2-3x-\lambda e^x$		$0.6 \to 0.4210489681$	2.000000000	*5
		$0.8 \to 0.3317744840$		4
		$1.0 \to 0.2575302854$		4
$f_5(x)=\cos(x)-x$	0.7390851332	$0.0 \to 1.570796327$	$\cos(x)$	1
&	(4)	$0.2 \to 1.306440008$	&	3
$H_5(x,\lambda)=$		$0.4 \to 1.110510504$	1.570796327	3
$\cos(x)-\lambda x$		$0.6 \to 0.958251898$		4
		$0.8 \to 0.837060789$		4
		$1.0 \to 0.739085133$		4

**oscillating values*

Table 2: Comparison of Classical Newton-Raphson (CNR) and Newton-homotopy using different start-systems (NHss).

Functions used: $f(x)=q(x)$ **= 'target system'**	**Root/s using NHss (Maple12);** $p(x)=0$ **= 'start-system'**	**Number of iterations for NHss & CNR; where** $\lambda = 0,0.2,0.4,0.6,0.8\,\&\,1.0$; $x_0 \to p(x)=0$
(f) $f_5(x)=x^2-(1-x)^5$; (See Javidi, 2009)	$H(x,\lambda)=(1-\lambda)p(x)-\lambda q(x)$	
	a) $p(x)=x^5-1$	a)1, 5, 6, 4, 6 & 5 ; (5)
	b) $p(x)=5x^4-1$	b)1, 10, 8, 8, 7& 8 ; (3)
	c) $p(x)=9x^2-1$	c)**1, 3, 2, 2, 2 & 3** ; (**3**)
	d) $p(x)=5x-1$	d)1, 3, 3, 3, 4 & 5 ; (4)
(g) $f_6(x)=x^3+4x^2-10$; (See Saeed & Khthr, 2010)	a) $p(x)=x^3-10$	a)1, 4, 4, 5, 5 & 6 ; (4)
	b) $p(x)=4x^2-10$	b)**1, 3, 3, 3, 4 & 4** ; (**6**)

In Table 2, it can be seen that the functions that use the start-system with lower degree of *n* (such as degree of two) converges faster than the higher degree.

4. CONCLUSIONS

As demonstrated in the numerical results, Newton-Homotopy using start-system method converges better or equal to the classical Newton-Raphson. It is also very important to have a proper start-system to ensure convergence is fast and computing time is reduced.

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