A Method of Computing Functions of Trapezoidal Fuzzy Variable and Its Application to Fuzzy Calculus

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Abstract

This paper introduces a method of computing functions of trapezoidal fuzzy variable. The method is based on the implementation of an unconstrained optimisation technique over the α*-cut of fuzzy interval. To show the effectiveness of the proposed method, we provide several numerical examples in computing the solutions of linear and non-linear fuzzy differential equations. The final results showed that the proposed method is capable to generate convex fuzzy solutions on time domain.*

Keywords: Fuzzy differential equations; trapezoidal fuzzy interval; unconstrained optimisation.

1. INTRODUCTION

In fuzzy set theory, the extension principle has been extensively used in many areas of discipline, namely in fuzzy optimisation problem (Shiang-Tai, 2004), fuzzy differential equation (Ma et al., 1999), fuzzy dynamical system (Ahmad and De Baets, 2009) and many more. Basically, the extension principle is the theoretical basis of fuzzy arithmetic, i.e., a process of extending real-valued functions to functions accepting fuzzy interval as arguments. The idea is very simple, but in practice, it is a complex problem.

Recently, Ahmad and Hasan (2011) have proposed an efficient computational method in order to compute functions accepting fuzzy interval as arguments. The proposed method is easy to implement and can be used in many practical problems. The convexity of fuzzy output is also guaranteed. In this paper, we will demonstrate the capability of the proposed method in order to approximate the solutions of linear and non-linear fuzzy differential equations.

2. THE EXTENSION PRINCIPLE

The extension principle (Zadeh, 1975a, Zadeh, 1975b) becomes an important tool in fuzzy set theory and applications. The idea is that each function $f : U \to V$ induces another function $f : F(U) \to F(V)$ defined for each fuzzy set A in U by

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & ,\text{if } y \in \text{range(f)}, \\ 0 & ,\text{if } y \notin \text{range(f)}. \end{cases} \tag{1}$$

If f is one to one mapping, we have

$$f(A)(y) = \begin{cases} A(f^{-1}(y)) & ,\text{if } y \in \text{range(f)}, \\ 0 & ,\text{if } y \notin \text{range(f)}. \end{cases} \tag{2}$$

To demonstrate the extension principle, we consider $f : R \to R$ defined by

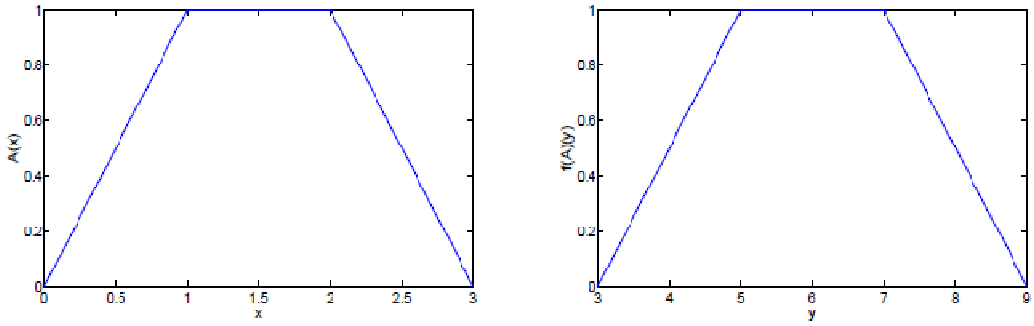
$$f(x) = 2x + 3$$

and the fuzzy interval $A \subseteq R$ (a fuzzy set defined on real line) is given by

$$A(x) = \begin{cases} 0 & ,\text{if } x < 0, \\ x & ,\text{if } 0 \le x \le 1, \\ 1 & ,\text{if } 1 \le x \le 2, \\ -x+3 & ,\text{if } 2 \le x \le 3, \\ 0 & ,\text{if } x > 3. \end{cases} \tag{3}$$

Then from Eq. (2), we can easily obtain $f : F(R) \to F(R)$ as follow:

$$f(A)(y)=\begin{cases} 0 & ,\text{if } y<3, \\ \dfrac{y-3}{2} & ,\text{if } 3\le y\le 5, \\ 1 & ,\text{if } 5\le y\le 7, \\ \dfrac{-y+9}{2} & ,\text{if } 7\le y\le 9, \\ 0 & ,\text{if } y>9. \end{cases} \tag{4}$$



(a) (b)

Figure 1: (a) The trapezoidal fuzzy interval A; (b) The output from the extension principle.

The graphs of (3) and (4) are depicted in Figs. 1(a) and 1(b), respectively. In general, if f is non-monotone function, then the calculation of (1) is not an easy task. This needs a contribution on developing an efficient computational algorithm for computing (1).

3. THE METHOD

Let $A=(a,b,c,d)$ be a trapezoidal fuzzy interval. The α-cut of A is denoted by $[A]^{\alpha}=[a_1^{\alpha},a_2^{\alpha}]$ for $\alpha\in(0,1]$, where $a_1^{\alpha}=a+\alpha(b-a)$ and $b_2^{\alpha}=d-\alpha(d-c)$. First, we discretise α in the form $\alpha_0<\alpha_1<\ldots<\alpha_{n-1}<\alpha_n$, where $\alpha_0=0$ and $\alpha_n=1$. The discretised α are equally spaced, that is $\alpha_j=\alpha_0+j\Delta h$, for $j=0,1,2,\ldots,n$ and $\Delta h=\frac{1}{n}>0$. In this study, Δh is called the discretisation spacing. After discretisation, we have a set of α with $(n+1)$ elements:

$$\alpha=\{\alpha_0,\ldots,\alpha_j,\ldots,\alpha_n\}. \tag{5}$$

This leads to a set of I with $(n+1)$ intervals:

$$I=\left\{[A]^{\alpha_0},\ldots,[A]^{\alpha_j},\ldots,[A]^{\alpha_n}\right\} \tag{6}$$

For the different α-cuts of A the following property holds (Möler and Reuter, 2007):

$$[A]^{\alpha_{j+1}}\subseteq[A]^{\alpha_j},\quad \forall\alpha_j,\alpha_{j+1}\in[0,1] \quad \text{with} \quad \alpha_j\le\alpha_{j+1}. \tag{7}$$

for $j=0,1,2,\ldots,n-1$.

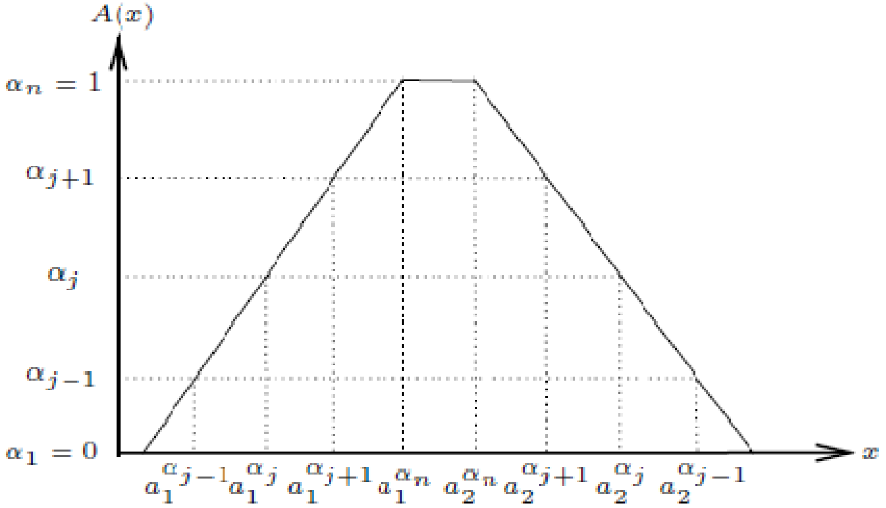


Figure 2. α-discretisation of a trapezoidal fuzzy interval (Ahmad and Hasan, 2011).

Since this property holds for all $\alpha \in [0,1]$, the α-cuts of A can be constructed as the union of sub-intervals as shown in the following equation (see Fig. 2):

$$[A]^{\alpha_j} = [a_1^{\alpha_j}, a_1^{\alpha_{j+1}}] \cup [a_1^{\alpha_{j+1}}, a_2^{\alpha_{j+1}}] \cup [a_2^{\alpha_{j+1}}, a_2^{\alpha_j}]. \tag{8}$$

Let $f : R \to R$ be a continuous function. Given a fuzzy interval A defined in R. In order to find $B = f(A)$ at each level of α_j for $j = 0,1,2,..., n$, we need to solve the following equations:

$$b_1^{\alpha_j} = \min\left[\min_{x\in[a_1^{\alpha_j}, a_1^{\alpha_{j+1}}]} f(x), \min_{x\in[a_1^{\alpha_{j+1}}, a_2^{\alpha_{j+1}}]} f(x), \min_{x\in[a_2^{\alpha_{j+1}}, a_2^{\alpha_j}]} f(x)\right], \tag{9}$$

$$b_2^{\alpha_j} = \max\left[\max_{x\in[a_1^{\alpha_j}, a_1^{\alpha_{j+1}}]} f(x), \max_{x\in[a_1^{\alpha_{j+1}}, a_2^{\alpha_{j+1}}]} f(x), \max_{x\in[a_2^{\alpha_{j+1}}, a_2^{\alpha_j}]} f(x)\right]. \tag{10}$$

Then, by using linear spline interpolation we interpolate the desired results $(b_1^{\alpha_j}, \alpha_j)$ as well as $(b_2^{\alpha_j}, \alpha_j)$ to obtain a fuzzy interval B. The next section, we will use this method in order to approximate the solution of fuzzy differential equations.

4. NUMERICAL EXAMPLES

In this section, we provide two numerical examples of computing the solution of linear and non-linear fuzzy differential equations. For this purpose, the method proposed in Section 3 will be incorporated into the generalised fuzzy Euler method proposed by Ahmad and Hasan (2010).

Example 1 *Consider the linear fuzzy differential equation:*

$$\begin{cases} Y'(t) = 2Y + 3, & t \in [0,1], \\ Y(0) = (1,2,3,4), \end{cases} \tag{11}$$

According to Ahmad et al. (2011), we need to solve the following system of ordinary differential equations:

$$\begin{cases} y_1'^{\alpha}(t) = 2y_1^{\alpha} + 3, & y_1^{\alpha}(0) = \alpha + 1, \\ y_2'^{\alpha}(t) = 2y_2^{\alpha} + 3, & y_2^{\alpha}(0) = -\alpha + 4. \end{cases} \tag{12}$$

The analytical solutions of (12) are

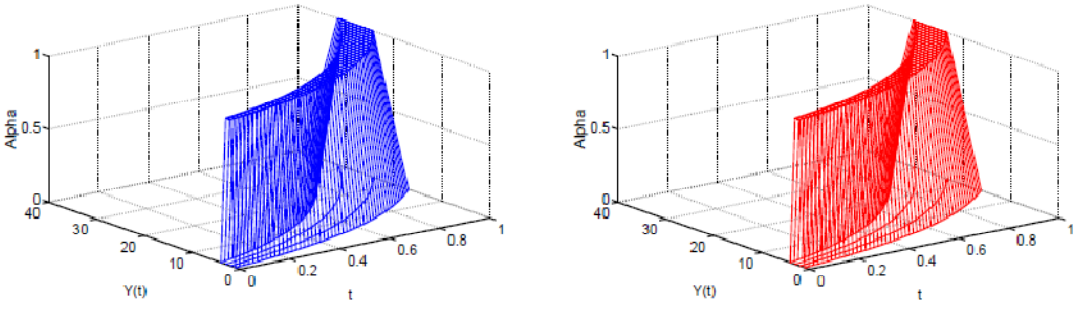
$$y_1^{\alpha}(t) = (5/2 + \alpha)e^{2t} - 3/2,$$

$$y_2^{\alpha}(t) = (11/2 - \alpha)e^{2t} - 3/2..$$

It is clear that

$$[Y(t)]^{\alpha} = [y_1^{\alpha}(t), y_2^{\alpha}(t)]$$

is the solution of fuzzy differential equation (11). It is illustrated in Fig. 3(a).



(a)

(b)

Figure 3. (a) The exact solution of Example 1; (b) The approximate solution of Example 1 with $h = 0.02$ and $N = 50$.

Using the generalised fuzzy Euler method (Ahmad and Hasan, 2010), the results are shown in Fig. 3(b). From the graph, we can see that the approximate solution converges to the exact solution. The numerical results of exact and approximate solution, and their errors at $t = 0.5$ are listed in Table 1.

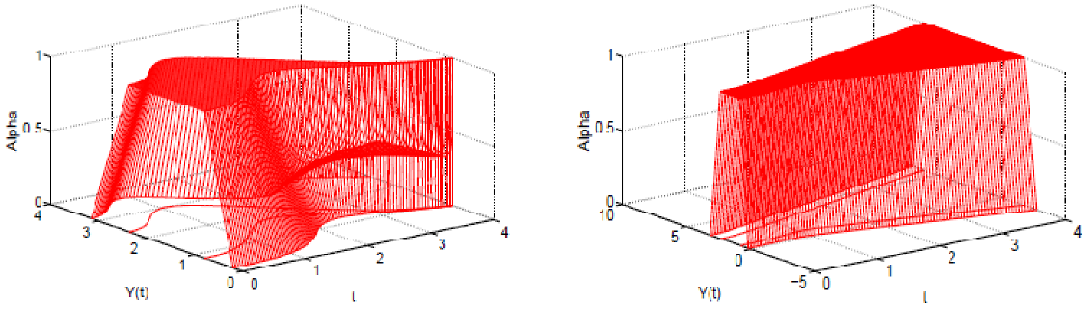
Table 1: The numerical results of Example 1 at $t = 0.5$

α	$y_1^{\alpha}(0.5)$	$y_{0.5,1}^{\alpha}$	$E_{0.5,\text{left}}^{\alpha}$	$y_2^{\alpha}(0.5)$	$y_{0.5,2}^{\alpha}$	$E_{0.5,\text{right}}^{\alpha}$
1.0	8.013986	7.830427	0.183559	10.732268	10.496263	0.236004
0.9	7.742158	7.563843	0.178314	11.004096	10.762847	0.241249
0.8	7.470330	7.297259	0.173070	11.275924	11.029430	0.246493
0.7	7.198501	7.030676	0.167825	11.547752	11.296014	0.251738
0.6	6.926673	6.764092	0.162581	11.819580	11.562598	0.256982
0.5	6.654845	6.497508	0.157336	12.091409	11.829181	0.262227
0.4	6.383017	6.230925	0.152091	12.363237	12.095765	0.267472
0.3	6.111189	5.964341	0.146847	12.635065	12.362348	0.272716
0.2	5.839360	5.697758	0.141602	12.906893	12.628932	0.277961
0.1	5.567532	5.431174	0.136358	13.178721	12.895516	0.283205
0.0	5.295704	5.164590	0.131113	13.450550	13.162099	0.288451

Example 2 *Consider the non-linear fuzzy differential equation:*

$$\begin{cases} Y'(t) = \sin(tY), \quad t \in [0,4], \\ Y(0) = (0.2, \pi/2, 3\pi/4, \pi). \end{cases} \tag{13}$$

Since the exact solution of (13) cannot be found analytically, we approximate its solution using the generalised fuzzy Euler method (Ahmad and Hasan, 2010). Using the step size $h = 0.04$ and $N = 100$, we obtain the approximate solution of (13) as plotted in Fig. 4(a). From the graph, we can see that the approximate solution of (13) is periodic as t increases. However, if we use the conventional fuzzy Euler method (Ma et al., 1999), the approximate solution of (13) is diverging as t increases (see Fig. 4(b)). This shows us that the generalised fuzzy Euler method is capable to generate periodic solution on the time domain.



(a) (b)

Figure 4: (a) The approximate solution of Example 2 using the method proposed by Ahmad and Hasan (2010); (b) The approximate solution of Example 2 using the method proposed by Ma et al. (1999).

5. CONCLUSIONS

In this paper, we have introduced a method of computing functions of a trapezoidal fuzzy variable. The method has two advantages: (a) the convexity of the output is ensured even if f is a non-monotone function; (b) it can be incorporated into any numerical method in order to approximate the solution of linear and non-linear fuzzy differential equations. This is a stronger requirement since there is no attention has been made, especially in solving non-linear fuzzy differential equations.

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