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A New Flexible Generalized Distribution: The Modified Kies Inverse Weibull Distribution with Properties, Estimations and Applications

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Abstract

This study introduces the Modified Kies Inverse Weibull (MKIW) distribution as a new generalisation of the Inverse Weibull (IW) distribution within the Modified Kies-G family of distributions. The proposed model extends existing IW generalisations through an odds-based exponential generator, thereby providing enhanced flexibility in modelling skewness, tail behaviour, and hazard rate structures. Several structural and statistical properties of the distribution are derived, including the quantile function, moments, skewness, kurtosis, median, mode, order statistics, and reliability measures. Parameter estimation is investigated using six estimation methods. A Monte Carlo simulation study is conducted to assess the performance of the estimators in terms of average bias, average absolute bias, mean relative error, standard deviation, and mean squared error. The results indicate that the maximum product of spacings and maximum likelihood methods provide the most efficient estimators across most parameter settings. The practical applicability of the MKIW distribution is illustrated using four real lifetime datasets. Comparative analyses show that the proposed model provides a competitive fit relative to the classical IW distribution and several of its extensions, including the exponentiated generalised, Kumaraswamy, Topp-Leone, and alpha power IW distributions. In particular, the MKIW distribution reduces the AIC by approximately 4.2% to 16.5% when compared with the classical IW model across the four datasets, demonstrating its flexibility and usefulness for modelling complex lifetime data.

1. Introduction

Statistical distributions provide the fundamental mathematical framework for modelling and analysing empirical data, enabling the quantification of uncertainty and variability inherent in natural and engineered processes. Thus, the selection of an appropriate statistical distribution is crucial, as it directly impacts the accuracy of parameter estimation, hypothesis testing, and decision-making processes in applied research [1]. Among the numerous statistical models, the Inverse Weibull (IW) distribution, introduced by [2], has attracted significant research attention due to its flexibility and practical relevance. Extensive studies have explored its theoretical properties and applications [3-18]. The IW distribution has also been employed as an alternative to the Weibull model in various fields, including wind speed modelling [19] and agricultural studies involving intuitionistic fuzzy sets [20], among several other practical applications.

Despite its applicability, the IW model often proves inadequate for datasets displaying pronounced skewness or multimodal tendencies, necessitating generalised forms to improve fit and interpretive power [21]. In response, several generalisations of the IW distribution have been proposed in the literature. For instance, the Marshall-Olkin Inverse Weibull (MOIW) distribution [22] extends the IW model by incorporating an additional tilt parameter through the Marshall-Olkin transformation, thereby enhancing tail flexibility; however, its hazard function is largely restricted to monotone or bathtub-shaped forms, limiting its suitability for datasets that exhibit unimodal (reversed bathtub) hazard behaviour. The Topp-Leone Inverse Weibull (TLIW) distribution [23] introduces a bounded-support generator that yields asymmetric density shapes, yet its structural complexity often imposes challenges in parameter identifiability and computational estimation, particularly for small to moderate sample sizes. Other notable generalisations include the Exponentiated Inverse Weibull [24], the Beta Inverse Weibull [25], and the Kumaraswamy Inverse Weibull [26] distributions, each of which augments the IW baseline through well-known families; nonetheless, these extensions typically involve two or more additional parameters, increasing model complexity without always yielding commensurate gains in goodness-of-fit across diverse lifetime datasets. A common limitation shared by many of these generalisations is that they do not simultaneously accommodate right-skewed, approximately symmetric, and L-shaped densities alongside increasing, decreasing, and reversed bathtub hazard shapes within a single parsimonious structure.

The Modified Kies Inverse Weibull (MKIW) distribution proposed in this study addresses these limitations by employing the Modified Kies-G (MK-G) family, introduced by [27], which incorporates only one additional shape parameter into the IW baseline. Unlike the MOIW and TLIW, the MK-G generator transforms the odds function $\frac{G(x)}{(1-G(x)}$ through an exponential structure governed by an additional shape parameter λ. This construction provides substantially greater flexibility in controlling skewness, kurtosis, tail weight, and hazard rate behaviour simultaneously. Consequently, the MKIW distribution can model L-shaped, right-skewed, and nearly symmetric densities, together with increasing, decreasing, and reversed bathtub hazard rate functions. Moreover, in contrast to multi-parameter extensions such as the Beta and Kumaraswamy IW distributions, the MKIW distribution maintains analytical tractability in its statistical properties, including closed-form expressions for the survival function, hazard function, and cumulative hazard function, as demonstrated in subsequent sections.

A significant advancement in the creation of more adaptable distributions is the development of the MK-G family of distributions, introduced by [27]. This family employs a versatile generalisation technique by introducing an additional shape parameter into a baseline distribution. The uniqueness of the MK-G family lies in its generator, which directly influences the skewness, kurtosis, and tail properties of the resulting distribution. This generator is renowned for its ability to produce distributions capable of modelling a rich variety of data shapes, including those that are J-shaped, U-shaped, left-skewed or right-skewed, thereby often providing a superior fit compared to the baseline model and other existing extensions.

A nonnegative random variable X is said to follow the IW distribution with α and β as the scale and shape parameters respectively, if its cumulative distribution function (CDF) and probability density function (PDF) take the form:

$$G(x;\alpha,\beta) = e^{-\left(\frac{\alpha}{x}\right)^{\beta}}, \tag{1}$$

$$g(x;\alpha,\beta) = \alpha^{\beta}\beta x^{-(\beta+1)}e^{-\left(\frac{\alpha}{x}\right)^{\beta}}. \tag{2}$$

The CDF and PDF of the Modified Kies-G family (MK-G) with a positive shape parameter λ are defined by

$$F(x;\lambda) = 1 - \exp\left[-\left(\frac{G(x)}{1-G(x)}\right)^{\lambda}\right], \qquad x \in \mathbb{R}^{+}, \tag{3}$$

and

$$f(x;\lambda) = \lambda\frac{g(x)G(x)^{\lambda-1}}{\left(1-G(x)\right)^{\lambda+1}}\exp\left[-\left(\frac{G(x)}{1-G(x)}\right)^{\lambda}\right], \tag{4}$$

where $G(x)$ and $g(x)$ are the CDF and PDF of a continuous baseline distribution.

The primary objective of this study is to introduce and examine a new distribution, the Modified Kies Inverse Weibull distribution, developed through the MK-G approach. Section 2 presents the proposed distribution and its reliability analysis, while Section 3 outlines its main mathematical and statistical properties. Section 4 describes six estimation techniques, and Section 5 reports the results of a Monte Carlo simulation assessing their performance. Section 6 applies the model to four real datasets, highlighting parameter estimates, discrimination measures, and goodness-of-fit results. Concluding remarks are provided in Section 7.

2. The Proposed Modified Kies Inverse Weibull Distribution

By applying the CDF and PDF of the IW distribution to the MK-G, the CDF and PDF for the MKIW distribution are respectively given by

$$F(x;\alpha,\beta,\lambda) = 1 - exp\left[-\left(\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right)^{\lambda}\right], \quad x > 0, \tag{5}$$

$$f(x;\alpha,\beta,\lambda) = \alpha^{\beta}\beta\lambda x^{-(\beta+1)}\frac{e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}}{\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{\lambda+1}} exp\left[-\left(\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right)^{\lambda}\right], \tag{6}$$

where $\alpha > 0$ is the scale parameter, while $\beta > 0$ and $\lambda > 0$ are shape parameters respectively. The survival function (SF), hazard function (HF), and cumulative hazard function (CHF) of the MKIW distribution are respectively given by

$$S(x;\alpha,\beta,\lambda) = 1 - F(x) = exp\left[-\left(\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right)^{\lambda}\right], \tag{7}$$

$$h(x;\alpha,\beta,\lambda) = \frac{f(x)}{S(x)} = \alpha^{\beta}\beta\lambda x^{-(\beta+1)}\frac{e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}}{\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{\lambda+1}}, \tag{8}$$

and

$$H(x;\alpha,\beta,\lambda) = -\,ln[S(x)] = \left(\frac{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right)^{\lambda}. \tag{9}$$

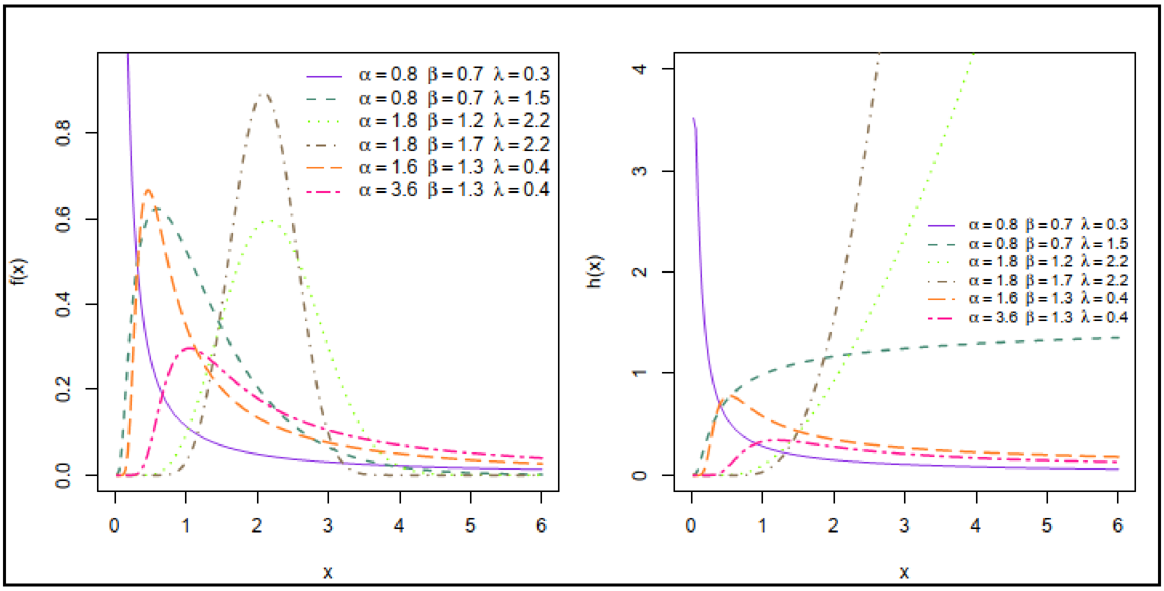

Fig. 1 *PDF (left panel) and HF (right panel) plots for the MKIW distribution*

The PDF and HF plots for the MKIW distribution under six parameter combinations are presented in Figure 1. Clearly, the density of MKIW distribution can be L-shaped, right-skewed and closely symmetric. The HF exhibits considerable flexibility, including increasing, decreasing and reversed bathtub shapes.

From Equation (8), the limiting behaviour of HF provides important structural insight into the MKIW distribution. As, $x \to 0^+$, we have, $\exp\left(-(\alpha/x)^\beta\right) \to 0$, implying that $h(x) \to 0$. Furthermore, as $x \to \infty$, $(\alpha/x)^\beta \to 0$ and $1 - \exp\left(-(\alpha/x)^\beta\right) \sim (\alpha/x)^\beta$, which yields $h(x) \sim \frac{\beta\lambda}{\alpha^{\beta\lambda}} x^{\beta\lambda - 1}$.

Therefore, the shape of the HF is primarily governed by the product $\beta\lambda$. Specifically, when $\beta\lambda < 1$, the HF exhibits a reversed bathtub shape, initially increasing from zero, attaining a unique maximum, and subsequently decreasing towards zero as $x \to \infty$. When $\beta\lambda = 1$, the HF increases and converges to a positive constant. In contrast, when $\beta\lambda > 1$, the HF increases monotonically without decline. The parameter α acts solely as a scale parameter and does not alter the qualitative behaviour of the HF. Consequently, the MKIW distribution provides a flexible yet analytically tractable framework for modelling diverse lifetime behaviours.

3. Mathematical and Statistical Properties

3.1 Linear Representation

The CDF and PDF of the MKIW distribution can be expressed as infinite series using the generalised binomial expansion, which facilitates the derivation of several statistical properties. For any real $\lambda > 0$,

$$(1 - z)^{-\lambda} = \sum_{j=0}^{\infty} \frac{\Gamma(\lambda + j)}{\Gamma(\lambda) j!} z^j, \quad |z| < 1.$$

Also, the exponential series expansion is given by

$$e^{-y} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k.$$

For the IW distribution, $G(x) = e^{-(\alpha/x)^\beta}$, $\quad 0 < G(x) < 1, x > 0$. The CDF of the MKIW distribution may be written as

$$F(x) = 1 - \exp\left\{-\left(\frac{G(x)}{1 - G(x)}\right)^\lambda\right\}.$$

Applying the exponential series expansion gives

$$F(x) = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{G(x)}{1 - G(x)}\right)^{\lambda k}.$$

Using the generalised binomial expansion for $(1 - G(x))^{-\lambda k}$, and separating the $k = 0$ term to avoid the singularity in $\Gamma(0)$, the CDF of the MKIW distribution can be represented as

$$F(x) = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} e^{-(\lambda k + j)(\alpha/x)^\beta}. \tag{10}$$

Differentiating Equation (10) with respect to x, the corresponding PDF is obtained as

$$f(x) = \beta\alpha^\beta x^{-\beta-1} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j) e^{-(\lambda k + j)(\alpha/x)^\beta}. \tag{11}$$

Equations (10) and (11) are valid for all positive real values of λ. Moreover, the convergence of the series is guaranteed since $0 < G(x) < 1$ for all $x > 0$, ensuring the absolute convergence of both the exponential and generalised binomial expansions.

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3.2 Moments

If $X \sim MKIW(\alpha, \beta, \lambda)$ with density function given in Equation (11), then the r-th ordinary moment is defined by

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x) dx.$$

Substituting Equation (11) into the above expression gives

$$\mu_r' = \beta\alpha^\beta \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j) \int_0^\infty x^{r-\beta-1} e^{-(\lambda k + j)(\alpha/x)^\beta} dx.$$

Let $y = \left(\frac{\alpha}{x}\right)^\beta$, then, $x = \alpha y^{-1/\beta}$, $\quad dx = -\frac{\alpha}{\beta} y^{-(1/\beta)-1} dy$.

Hence,

$$\int_0^\infty x^{r-\beta-1} e^{-(\lambda k + j)(\alpha/x)^\beta} dx = \frac{\alpha^{r-\beta}}{\beta} \int_0^\infty y^{-r/\beta} e^{-(\lambda k + j) y} dy.$$

Using the Gamma integral

$$\int_0^\infty y^{a-1} e^{-by} dy = \frac{\Gamma(a)}{b^a}, \quad a > 0, \; b > 0,$$

Yields

$$\int_0^\infty y^{-r/\beta} e^{-(\lambda k + j) y} dy = \Gamma\left(1 - \frac{r}{\beta}\right) (\lambda k + j)^{-(1-r/\beta)}.$$

Therefore, the rth moment of the MKIW distribution is obtained as

$$\mu_r' = \alpha^r \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j)^{r/\beta}, \quad r < \beta. \tag{12}$$

The condition $r < \beta$ is required to ensure the existence of the Gamma function term $\Gamma(1 - r/\beta)$. The first four ordinary moments of the MKIW distribution are therefore given respectively by

$$\mu_1' = \alpha \Gamma\left(1 - \frac{1}{\beta}\right) \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j)^{1/\beta},$$

$$\mu_2' = \alpha^2 \Gamma\left(1 - \frac{2}{\beta}\right) \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j)^{2/\beta},$$

$$\mu_3' = \alpha^3 \Gamma\left(1 - \frac{3}{\beta}\right) \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j)^{3/\beta},$$

and

$$\mu_4' = \alpha^4 \Gamma\left(1 - \frac{4}{\beta}\right) \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j)^{4/\beta}.$$

The numerical computation of the first four moments of the MKIW distribution for selected values of the parameters is shown in Table 1. It can be seen that the moments of the MKIW increase with higher values of α and λ, indicating their strong influence on the distribution's dispersion and shape respectively. Conversely, larger β values lead to smaller moment values, reflecting the effect of β in controlling the tail behaviour and concentration of the distribution.

Table 1 *The first, second, third and fourth moments for different parameter values of MKIW distribution*

α	β	λ	μ'_1	μ'_2	μ'_3	μ'_4	α	β	λ	μ'_1	μ'_2	μ'_3	μ'_4
0.3	4.5	0.1	0.2315	0.1051	0.0843	0.1587	1.4	4.5	0.1	1.0803	2.2896	8.5632	75.271
0.3	4.5	0.2	0.3214	0.1582	0.1250	0.2247	1.4	4.5	0.2	1.5001	3.4451	12.700	106.59
0.3	4.5	0.3	0.3437	0.1546	0.1033	0.1540	1.4	4.5	0.3	1.6040	3.3662	10.495	73.030
0.3	5.5	0.1	0.2269	0.0849	0.0458	0.0370	1.4	5.5	0.1	1.0590	1.8486	4.6549	17.549
0.3	5.5	0.2	0.3069	0.1257	0.0690	0.0542	1.4	5.5	0.2	1.4321	2.7378	7.0145	25.722
0.3	5.5	0.3	0.3299	0.1284	0.0629	0.0426	1.4	5.5	0.3	1.5394	2.7964	6.3905	20.193
0.3	6.5	0.1	0.2258	0.0764	0.0333	0.0191	1.4	6.5	0.1	1.0536	1.6643	3.3861	9.0548
0.3	6.5	0.2	0.2993	0.1110	0.0502	0.0286	1.4	6.5	0.2	1.3966	2.4163	5.1028	13.542
0.3	6.5	0.3	0.3222	0.1160	0.0485	0.0246	1.4	6.5	0.3	1.5037	2.5259	4.9324	11.676
2.6	4.5	0.1	2.0063	7.8967	54.849	895.38	3.7	4.5	0.1	2.8552	15.992	158.07	3672.2
2.6	4.5	0.2	2.7859	11.882	81.345	1267.9	3.7	4.5	0.2	3.9645	24.063	234.43	5200.0
2.6	4.5	0.3	2.9788	11.610	67.222	868.76	3.7	4.5	0.3	4.2391	23.512	193.73	3563.0
2.6	5.5	0.1	1.9667	6.3758	29.816	208.76	3.7	5.5	0.1	2.7988	12.912	85.927	856.15
2.6	5.5	0.2	2.6596	9.4426	44.930	305.97	3.7	5.5	0.2	3.7848	19.123	129.49	1254.9
2.6	5.5	0.3	2.8589	9.6446	40.933	240.21	3.7	5.5	0.3	4.0684	19.532	117.97	985.14
2.6	6.5	0.1	1.9568	5.7403	21.689	107.71	3.7	6.5	0.1	2.7846	11.625	62.506	441.75
2.6	6.5	0.2	2.5936	8.3337	32.685	161.09	3.7	6.5	0.2	3.6909	16.877	94.196	660.66
2.6	6.5	0.3	2.7926	8.7118	31.593	138.89	3.7	6.5	0.3	3.9741	17.643	91.049	569.63

3.3 The Moment Generating Function

The moment generating function (MGF) of a random variable $X \sim MKIW(\alpha, \beta, \lambda)$ is defined as

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)dx, \tag{13}$$

provided that the integral converges.
Substituting Equation (11) into Equation (13) yields

$$M_X(t) = \beta\alpha^\beta \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\Gamma(\lambda k + j)}{\Gamma(\lambda k) j!} (\lambda k + j) \int_0^\infty x^{-\beta-1} e^{tx} e^{-(\lambda k+j)(\alpha/x)^\beta} dx. \tag{14}$$

Let $y = \left(\frac{\alpha}{x}\right)^\beta$, then, $x = \alpha y^{-1/\beta}$, $\quad dx = -\frac{\alpha}{\beta} y^{-(1/\beta)-1} dy$.

Hence,

$$\int_0^\infty x^{-\beta-1} e^{tx} e^{-(\lambda k+j)(\alpha/x)^\beta} dx = \frac{\alpha^{-\beta}}{\beta} \int_0^\infty y\, e^{t\alpha y^{-1/\beta}} e^{-(\lambda k+j)y} dy.$$

Expanding the exponential term,

$$e^{t\alpha y^{-1/\beta}} = \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} y^{-n/\beta},$$

gives

$$\int_0^\infty y\, e^{t\alpha y^{-1/\beta}} e^{-(\lambda k+j)y} dy = \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} \int_0^\infty y^{1-n/\beta} e^{-(\lambda k+j)y} dy.$$

Using the Gamma integral formula,

$$\int_0^\infty y^{a-1} e^{-by} dy = \frac{\Gamma(a)}{b^a}, \quad a > 0,\ b > 0,$$

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yields

$$\int_0^\infty y^{1-n/\beta}\, e^{-(\lambda k+j)y} dy = \Gamma\left(2-\frac{n}{\beta}\right)(\lambda k+j)^{-(2-n/\beta)}.$$

Therefore,

$$\int_0^\infty x^{-\beta-1}\, e^{tx} e^{-(\lambda k+j)(\alpha/x)^\beta} dx = \frac{\alpha^{-\beta}}{\beta}\sum_{n=0}^{\infty}\frac{(t\alpha)^n}{n!}\Gamma\left(2-\frac{n}{\beta}\right)(\lambda k+j)^{-(2-n/\beta)}. \tag{15}$$

Substituting Equation (15) into Equation (14), the MGF of the MKIW distribution is obtained as

$$M_X(t) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+1}}{k!}\frac{\Gamma(\lambda k+j)}{\Gamma(\lambda k)j!}\frac{(t\alpha)^n}{n!}\Gamma\left(2-\frac{n}{\beta}\right)(\lambda k+j)^{n/\beta-1}, \tag{16}$$

subject to the condition $n < 2\beta$, which guarantees the existence of the Gamma function term $\Gamma(2-n/\beta)$.

3.4 Distribution of Order Statistics

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics obtained from a random sample of size n drawn from the MKIW distribution. The PDF and CDF of the i-th order statistic $X_{(i)}$ are respectively given by

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!\,(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x), \tag{17}$$

and

$$F_{X_{(i)}}(x) = \sum_{j=i}^{n}\binom{n}{j}[F(x)]^j[1-F(x)]^{n-j}, \tag{18}$$

where $F(x)$ and $f(x)$ denote the CDF and PDF of the MKIW distribution, respectively. Substituting Equations (10) and (11) into Equation (17), the PDF of the i-th order statistic becomes

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!\,(n-i)!}\beta\alpha^\beta x^{-\beta-1}[S(x)]^{i-1}[1-S(x)]^{n-i}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+1}}{k!}\frac{\Gamma(\lambda k+j)}{\Gamma(\lambda k)j!}(\lambda k + j)e^{-(\lambda k+j)(\alpha/x)^\beta}, \tag{19}$$

where

$$S(x) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+1}}{k!}\frac{\Gamma(\lambda k+j)}{\Gamma(\lambda k)j!}e^{-(\lambda k+j)(\alpha/x)^\beta}.$$

Similarly, substituting Equation (10) into Equation (18), the CDF of the i-th order statistic is obtained as

$$F_{X_{(i)}}(x) = \sum_{r=i}^{n}\binom{n}{r}[S(x)]^r[1-S(x)]^{n-r}. \tag{20}$$

The representation in Equation (20) is computationally more efficient than the direct nested expansion form since it avoids repeated infinite summations raised to powers. For recursive computation, define

$$R_0(x) = 1, \quad R_r(x) = S(x)R_{r-1}(x), \quad r \geq 1.$$

Then Equation (20) may equivalently be expressed as

$$F_{X_{(i)}}(x) = \sum_{r=i}^{n}\binom{n}{r}R_r(x)[1-S(x)]^{n-r},$$

which provides a numerically stable recursive structure for evaluating the order statistic distribution. For special cases, the minimum and maximum order statistics respectively reduce to

$$F_{X_{(1)}}(x) = 1 - [1 - S(x)]^n, \tag{21}$$

and

$$F_{X_{(n)}}(x) = [S(x)]^n. \tag{22}$$

These simplified forms follow directly from the general order statistic distribution in Equation (20).

3.5 Quantile Function and Median

The quantile function (QF) of a distribution is the inverse of its CDF. From Equation (5), the QF of MKIK distribution is given by

$$x_u = \alpha \left[log \left(1 + \{-\,ln(1-u)\}^{-\frac{1}{\lambda}} \right) \right]^{-\frac{1}{\beta}}, \;\; 0 < u < 1, \tag{23}$$

setting $u = 0.5$, MKIW's median can be obtained as

$$\text{Med}(x) = x_{0.5} = \alpha \left[log \left(1 + (ln\,2)^{\frac{-1}{\lambda}} \right) \right]^{-\frac{1}{\beta}} \tag{24}$$

3.6 Skewness and Kurtosis

The skewness of the MKIW distribution, measured via Bowley's quartile-based coefficient [28], and the kurtosis, assessed through Moors' octile-based index [29] using the QF defined in Equation (23), are respectively given by

$$S_B = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}}, \tag{25}$$

and

$$K_M = \frac{x_{0.875} - x_{0.625} + x_{0.375} - x_{0.125}}{x_{0.75} - x_{0.25}}. \tag{26}$$

The graphical representations of these measures are displayed in Figure 2. The top (at $\lambda = 2.5$) and bottom (at $\lambda = 3.5$) panels reveal the behaviour of S_B and K_M across $\alpha, \beta \in (0.5, 1.5)$ respectively. The skewness plots indicate that the distribution is predominantly right-skewed at $\lambda = 0.5$, while at $\lambda = 3.5$ the skewness varies between negative and positive values. For both λ values, the kurtosis remains below 3, confirming a platykurtic distribution.

3.7 Mode

Mode of the distribution can be obtained as a solution of the equation,

$$\frac{d}{dx} ln[f(x)] = 0. \tag{27}$$

Substituting Equation (6) into (27), the mode of the MKIW distribution is given by

$$\lambda + (\lambda+1)\frac{e^{-(\alpha/x)^\beta}}{1 - e^{-(\alpha/x)^\beta}} - \frac{\lambda}{1 - e^{-(\alpha/x)^\beta}} \left(\frac{e^{-(\alpha/x)^\beta}}{1 - e^{-(\alpha/x)^\beta}} \right)^{\lambda} - \frac{\beta+1}{\beta} \left(\frac{\alpha}{x} \right)^{\beta} = 0 \tag{28}$$

The expression in Equation (28) has no closed-form solution; therefore, the mode is determined numerically for specified values of α, β, and λ. The numerical computation for the mode of the MKIW distribution for some selected parameter values is presented in Table 2. The numerical results indicate that the mode increases monotonically with α, β, and λ. While α indices a near-linear scaling effect, both β, and λ act as shape parameters that shift the peak of the distribution rightwards. This confirms the strong flexibility of the MKIW model in controlling modal location.

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Table 2 *Mode of the MKIW distribution for some selected parameter values*

α	β	λ	Mode	α	β	λ	mode	α	β	λ	mode
1.5	0.8	0.1	0.0274	1.2	0.1	0.7	0.0001	0.3	1.2	1.5	0.3063
1.5	0.8	0.4	0.1562	1.2	0.4	0.7	0.0207	0.4	1.2	1.5	0.4084
1.5	0.8	0.8	0.4272	1.2	0.8	0.7	0.2746	0.8	1.2	1.5	0.8167
1.5	0.8	1.2	0.8909	1.2	1.2	0.7	0.5427	1.2	1.2	1.5	1.2251
1.5	0.8	1.6	1.3742	1.2	1.6	0.7	0.7252	1.6	1.2	1.5	1.6334
1.5	0.8	1.8	1.5568	1.2	1.8	0.7	0.7911	1.8	1.2	1.5	1.8376
1.5	0.8	2.2	1.8119	1.2	2.2	0.7	0.8888	2.2	1.2	1.5	2.2460
1.5	0.8	2.5	1.9365	1.2	2.5	0.7	0.9410	2.5	1.2	1.5	2.5523
1.5	0.8	2.9	2.0490	1.2	2.9	0.7	0.9923	2.9	1.2	1.5	2.9606
1.5	0.8	3.6	2.1645	1.2	3.6	0.7	1.0512	3.6	1.2	1.5	3.6752

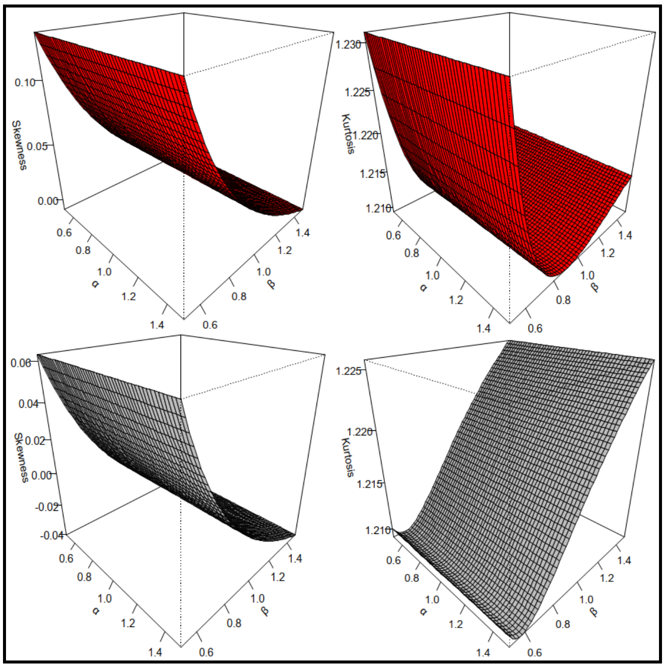

Fig. 2 *Plots for the Bowley's skewness and Moors' kurtosis of MKIW distribution*

4. Illustrations Estimation Methods

The estimation of the unknown parameters of the MKIW distribution are investigated using different approaches. Hereafter, let $x_i = x_1, x_2, \dots, x_n$ be a random sample of n observations and $x_{(i)} = x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be their values in ascending order; taken from the random variable $X \sim \text{MKIW}(\boldsymbol{\psi})$, where $\boldsymbol{\psi} = (\alpha, \beta, \lambda)$.

4.1 Maximum Likelihood (ML) Estimation

The ML estimates (MLEs) of α, β, and λ can be obtained by maximizing, with respect to α, β, and λ, the log-likelihood function for $\boldsymbol{\psi}$ is expressed by

$$\ell(\boldsymbol{\psi}) = \sum_{i=1}^{n}\left[\beta \ln \alpha + \ln \beta + \ln \lambda - (\beta+1)\ln x_i - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta} - (\lambda+1)\ln\left(1 - e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}\right) - \left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1 - e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right)^{\lambda}\right], \quad (29)$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{\beta}{\alpha}\left[1 + \left(\frac{\alpha}{x_i}\right)^{\beta}\left(-\lambda - (\lambda+1)\left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right) + \frac{\lambda}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right)^{\lambda}\right)\right] = 0, \tag{30}$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n}\left[\ln\alpha + \frac{1}{\beta} - \ln x_i - \left(\frac{\alpha}{x_i}\right)^{\beta} \ln\left(\frac{\alpha}{x_i}\right)\left\{\lambda + (\lambda+1)\left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right) - \frac{\lambda}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right)^{\lambda}\right\}\right] = 0 \tag{31}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n}\left[\frac{1}{\lambda} - \left(\frac{\alpha}{x_i}\right)^{\beta} - \ln\left(1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}\right) - \left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right)^{\lambda} \ln\left(\frac{e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right)\right] = 0 \tag{32}$$

These equations are nonlinear and coupled, and generally require numerical optimization (e.g., Newton-Raphson, or quasi-Newton) to solve for the MLEs of $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$.

4.2 Maximum Product of Spacings (MPS) Estimation

The MPS estimates; $\hat{\alpha}_{MPS}$, $\hat{\beta}_{MPS}$, and $\widehat{\lambda}_{MPS}$, of α, β, and λ are respectively obtained by maximizing:

$$M = \frac{1}{n+1} + \sum_{i=1}^{n+1} \log\left[\left(1-e^{-\left(\frac{e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}\right)^{\lambda}}\right) - \left(1-e^{-\left(\frac{e^{-\left(\frac{\alpha}{x_{(i-1)}}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_{(i-1)}}\right)^{\beta}}}\right)^{\lambda}}\right)\right]^{2}. \tag{33}$$

4.3 Cramér-von Mises (CVM) Minimum Distance Estimation

The CVM estimates; $\hat{\alpha}_{CVM}$, $\hat{\beta}_{CVM}$, and $\widehat{\lambda}_{CVM}$, of α, β, and λ are respectively obtained by minimizing:

$$C = \sum_{i=1}^{n}\left[1-e^{-\left(\frac{e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}\right)^{\lambda}} - \frac{2i-1}{2n}\right]^{2} + \frac{1}{12n}. \tag{34}$$

4.4 Anderson-Darling (AD) Estimation

The AD estimates; $\hat{\alpha}_{AD}$, $\hat{\beta}_{AD}$, and $\widehat{\lambda}_{AD}$, of α, β, and λ are respectively obtained by minimizing:

$$C = \sum_{i=1}^{n}\left[1-e^{-\left(\frac{e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}{1-e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}\right)^{\lambda}} - \frac{2i-1}{2n}\right]^{2} + \frac{1}{12n}. \tag{35}$$

4.5 Ordinary Least Square (OLS) Estimation

The OLS estimates; $\hat{\alpha}_{OLS}$, $\hat{\beta}_{OLS}$, and $\widehat{\lambda}_{OLS}$, of α, β, and λ are respectively obtained by minimizing:

$$L = \sum_{i=1}^{n} \left[1 - e^{-\left(\frac{e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}{1 - e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}} \right)^{\lambda}} - \frac{i}{n+1} \right]^2 . \tag{36}$$

4.6 Weighted Least Square (WLS) Estimation

The WLS estimates; $\hat{\alpha}_{\mathrm{WLS}}$, $\hat{\beta}_{\mathrm{WLS}}$, and $\widehat{\lambda}_{\mathrm{WLS}}$, of α, β, and λ are respectively obtained by minimizing:

$$W = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \sum_{i=1}^{n} \left[1 - e^{-\left(\frac{e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}}{1 - e^{-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}}} \right)^{\lambda}} - \frac{i}{n+1} \right]^2 . \tag{37}$$

Equations (33) - (37) involve non-linear functions, hence, require numerical optimization techniques to minimize or maximize the respective objective functions. In this study, the R version 4.5.2 was employed.

The six methods were selected to represent three classical estimation paradigms, while Bayesian and method of moments were excluded to avoid prior sensitivity, computational intensity, and lack of closed-form solutions for three-parameter distributions. MLE serves as the standard benchmark due to its asymptotic efficiency [30]. MPSE is included as a robust alternative that often outperforms MLE for distributions with unbounded likelihoods or heavy tails [31]. ADE and CVME represent minimum distance estimators, offering robustness against outliers and strong finite-sample performance [32] and [33]. The OLSE and WLSE provide computationally simple least-squares baselines based on linearised distributional forms [34]. The method of moments was omitted as it frequently fails to produce unique or closed-form solutions for multi-parameter distributions [35].

5. Simulation Study

A Monte Carlo simulation study was conducted to examine the finite sample performance of the six distinct estimators for the parameters of the MKIW distribution. The simulation experiment was designed to assess the consistency, accuracy and stability of the estimators under different sample sizes. Random samples were generated from the MKIW distribution through the inversion method by employing its quantile function, which is given by:

$$x_i = \alpha \left[log \left(1 + \{ - ln(1 - u_i) \}^{-\frac{1}{\lambda}} \right) \right]^{-\frac{1}{\beta}}, i = 1,2, \dots, n, \qquad 0 < u < 1, \tag{38}$$

where u follows a standard uniform distribution, $U(0,1)$.

The four parameter combinations, $\boldsymbol{\psi}_i = (\alpha, \beta, \lambda)$, considered in the study were $\boldsymbol{\psi}_1 = (0.7, 0.5, 0.2)$, $\boldsymbol{\psi}_2 = (1.1, 0.8, 1.2)$, $\boldsymbol{\psi}_3 = (0.9, 0.3, 0.6)$, and $\boldsymbol{\psi}_4 = (0.3, 0.2, 0.5)$. The simulation was performed for sample sizes $n = 25, 50, 100, 150, 200$, and 250. For each configuration, $R = 1000$ Monte Carlo replications were generated. For each simulated sample, the unknown parameters were estimated using the ADE, CVME, MLE, MPSE, OLSE, and WLSE methods via the `L-BFGS-B` optimisation algorithm implemented in R software. Positive lower bounds were imposed on the parameters during optimisation to ensure numerical stability and admissible parameter estimates. The performance of the estimators was evaluated using the following criteria: Average bias (Bias), Average absolute error (AAE), Mean relative error (MRE), Standard deviation (SD), and Mean squared error (MSE).

$$\text{Bias} = \frac{1}{R}\sum_{i=1}^{R}\left(\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}\right), \quad \text{AAE} = \frac{1}{R}\sum_{i=1}^{R}\left|\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}\right|, \quad \text{MRE} = \frac{1}{R}\sum_{i=1}^{R}\left|\frac{\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}}{\boldsymbol{\psi}}\right|,$$

$$\text{SD} = \sqrt{\frac{1}{R-1}\sum_{i=1}^{R}\left(\widehat{\boldsymbol{\psi}}_i - \overline{\widehat{\boldsymbol{\psi}}}\right)^2}, \quad \text{and} \quad \text{MSE} = \frac{1}{R}\sum_{i=1}^{R}\left(\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}\right)^2,$$

where $\boldsymbol{\psi} = (\alpha, \beta, \lambda)$.

The average bias, AAE, MRE, SD, and MSE values are presented in Tables 3 to 7 respectively, with the corresponding graphical representations shown in Figures 3 to 6.
The results of simulations are presented in Tables 3 to 7, while the graphical representations of these tables, which correspond to the numerical results of simulations, are displayed in Figures 3 to 6, respectively. Based on a careful examination of the Tables and the plots, the following observations are offered:

i. Consistency. All six estimators (ADE, CVME, MLE, MPSE, OLSE, WLSE) demonstrate statistical consistency. For every parameter configuration, the bias, AAE, MRE, SD, and MSE decrease as the sample size increases from 25 to 250.
ii. The MPSE is the overall best performing method. It consistently yields the smallest or near-smallest bias, MSE, and SD across all sample sizes and parameter configurations, particularly when the true λ is moderate to large (e.g., 1.2).
iii. The OLSE and the CVME are the least performing. They produce the largest bias, AAE, and MSE for almost all scenarios, especially for the parameter λ. Their poor performance persists even at $n = 250$.
iv. The MLE performs nearly as well as MPSE for simpler configurations such as MKIW (0.7, 0.5, 0.2) and MKIW $(0.3, 0.2, 0.5)$. However, MLE becomes noticeably less stable than MPSE when λ is large ($\lambda = 1.2$) or when bias for β and λ is examined jointly.
v. The Parameter λ is the most difficult to estimate, especially when λ is large ($\lambda = 1.2$). Across all tables and all methods, estimates for λ show substantially larger bias, MSE, and SD than estimates for α and β. This pattern holds regardless of the true parameter combination.
vi. The ADE and WLSE as intermediate options. The ADE and WLSE rank between MPSE and the poor performers. They may serve as reasonable robust alternatives if MPSE is computationally prohibitive, though they never outperform MPSE in any configuration.
vii. For researchers working with the MKIW distribution, the MPSE is strongly recommended as the default estimation technique. The OLSE and CVME may be avoided. The MLE is a viable second choice when λ is known to be small.

Table 3 *Simulation results of bias of the estimates for the MKIW distribution*

n	ADE (α, β, λ)	CVME (α, β, λ)	MLE (α, β, λ)	MPSE (α, β, λ)	OLSE (α, β, λ)	WLSE (α, β, λ)
			MKIW (α, β, λ) = MKIW (0.7, 0.5, 0.2)			
25	(0.274, 0.077, 0.011)	(0.729, 0.044, 0.054)	(0.380, 0.040, 0.012)	(0.831, -0.030, 0.045)	(0.253, 0.038, 0.083)	(0.496, 0.054, 0.038)
50	(0.077, 0.036, 0.005)	(0.263, 0.022, 0.124)	(0.104, 0.015, -0.002)	(0.426, -0.018, 0.023)	(0.102, 0.020, 0.030)	(0.175, 0.031, 0.014)
100	(0.027, 0.018, 0.001)	(0.087, 0.013, 0.007)	(0.030, 0.007, -0.006)	(0.161, -0.005, 0.011)	(0.022, 0.013, 0.011)	(0.052, 0.019, 0.005)
150	(0.033, 0.008, 0.002)	(0.093, 0.004, 0.007)	(0.023, 0.003, -0.006)	(0.091, 0.000, 0.006)	(0.049, 0.004, 0.009)	(0.063, 0.009, 0.005)
200	(0.028, 0.007, 0.001)	(0.064, 0.004, 0.004)	(0.010, 0.002, -0.005)	(0.077, -0.002, 0.006)	(0.033, 0.004, 0.006)	(0.042, 0.008, 0.003)
250	(0.008, 0.009, -0.001)	(0.030, 0.009, 0.001)	(0.000, 0.003, -0.006)	(0.052, 0.000, 0.004)	(0.006, 0.009, 0.002)	(0.010, 0.012, 0.000)
			MKIW (α, β, λ) = MKIW (1.1, 0.8, 1.2)			
25	(-0.079, 0.203, 1.004)	(-0.179, 0.210, 1.319)	(-0.033, 0.231, 0.418)	(-0.082, 0.068, 1.080)	(-0.356, 0.029, 2.064)	(-0.135, 0.202, 0.956)
50	(-0.028, 0.094, 0.329)	(-0.136, 0.065, 0.879)	(-0.024, 0.087, 0.169)	(-0.031, 0.036, 0.366)	(-0.237, -0.032, 1.258)	(-0.061, 0.090, 0.415)
100	(-0.018, 0.043, 0.111)	(-0.093, 0.021, 0.468)	(-0.013, 0.045, 0.047)	(0.002, 0.050, 0.075)	(-0.151, -0.034, 0.658)	(-0.038, 0.044, 0.193)
150	(-0.012, 0.022, 0.040)	(-0.069, 0.006, 0.303)	(-0.008, 0.030, 0.022)	(0.010, 0.050, 0.006)	(-0.107, -0.031, 0.419)	(-0.025, 0.023, 0.097)
200	(-0.007, 0.018, 0.034)	(-0.059, -0.010, 0.258)	(-0.009, 0.021, 0.020)	(0.008, 0.044, -0.005)	(-0.088, -0.039, 0.340)	(-0.019, 0.014, 0.087)
250	(-0.003, 0.017, 0.017)	(-0.038, 0.005, 0.158)	(-0.005, 0.021, 0.009)	(0.012, 0.045, -0.022)	(-0.060, -0.018, 0.216)	(-0.008, 0.020, 0.038)
			MKIW (α, β, λ) = MKIW (0.9, 0.3, 0.6)			
25	(0.110, 0.048, 0.058)	(0.072, 0.026, 0.279)	(0.131, 0.028, 0.085)	(0.162, 0.031, 0.138)	(-0.256, -0.008, 0.475)	(0.092, 0.037, 0.162)
50	(0.045, 0.019, 0.024)	(-0.003, 0.008, 0.155)	(0.069, -0.002, 0.032)	(0.146, 0.020, 0.047)	(-0.164, -0.009, 0.237)	(0.028, 0.017, 0.068)
100	(0.042, 0.008, 0.011)	(0.017, 0.002, 0.058)	(0.013, -0.003, 0.003)	(0.084, 0.022, 0.002)	(-0.066, -0.006, 0.088)	(0.035, 0.008, 0.025)
150	(0.021, 0.007, 0.002)	(0.003, 0.001, 0.034)	(0.005, -0.004, 0.001)	(0.055, 0.017, -0.004)	(-0.051, -0.004, 0.052)	(0.019, 0.006, 0.012)
200	(0.011, 0.003, 0.009)	(0.000, 0.000, 0.028)	(0.007, -0.003, -0.002)	(0.033, 0.016, -0.005)	(-0.039, -0.004, 0.041)	(0.009, 0.004, 0.015)
250	(0.006, 0.006, -0.002)	(-0.003, 0.005, 0.011)	(-0.002, -0.003, -0.003)	(0.042, 0.014, -0.008)	(-0.034, 0.002, 0.021)	(0.003, 0.007, 0.001)
			MKIW (α, β, λ) = MKIW (0.3, 0.2, 0.5)			
25	(0.228, 0.030, 0.047)	(0.262, 0.014, 0.208)	(0.320, 0.004, 0.058)	(0.338, 0.018, 0.080)	(-0.007, -0.002, 0.310)	(0.247, 0.024, 0.119)
50	(0.073, 0.013, 0.011)	(0.075, 0.003, 0.092)	(0.111, -0.003, 0.023)	(0.176, 0.014, 0.026)	(-0.022, -0.005, 0.128)	(0.079, 0.009, 0.037)
100	(0.036, 0.004, 0.013)	(0.034, 0.000, 0.047)	(0.028, -0.002, 0.003)	(0.099, 0.013, -0.001)	(-0.010, -0.004, 0.065)	(0.036, 0.004, 0.024)
150	(0.022, 0.004, 0.005)	(0.022, 0.002, 0.027)	(0.023, -0.003, 0.000)	(0.057, 0.011, -0.005)	(-0.006, -0.001, 0.033)	(0.025, 0.005, 0.009)
200	(0.016, 0.004, -0.001)	(0.017, 0.004, 0.007)	(0.014, -0.002, -0.002)	(0.037, 0.010, -0.005)	(-0.004, 0.002, 0.015)	(0.017, 0.005, 0.002)
250	(0.017, 0.002, 0.003)	(0.017, 0.001, 0.018)	(0.007, -0.002, -0.003)	(0.039, 0.008, -0.006)	(0.000, 0.000, 0.020)	(0.016, 0.003, 0.007)

Table 4 *Simulation results of AAE of the estimates for the MKIW distribution*

n	ADE (α, β, λ)	CVME (α, β, λ)	MLE (α, β, λ)	MPSE (α, β, λ)	OLSE (α, β, λ)	WLSE (α, β, λ)
			MKIW (α, β, λ) = MKIW (0.7, 0.5, 0.2)			
25	(0.702, 0.149, 0.065)	(1.142, 0.168, 0.104)	(0.847, 0.131, 0.066)	(1.021, 0.104, 0.068)	(0.779, 0.168, 0.123)	(0.905, 0.153, 0.085)
50	(0.414, 0.092, 0.043)	(0.588, 0.114, 0.161)	(0.452, 0.079, 0.039)	(0.592, 0.074, 0.041)	(0.499, 0.114, 0.062)	(0.511, 0.102, 0.049)
100	(0.272, 0.059, 0.028)	(0.347, 0.074, 0.035)	(0.275, 0.052, 0.025)	(0.302, 0.051, 0.027)	(0.322, 0.074, 0.036)	(0.315, 0.067, 0.031)
150	(0.225, 0.047, 0.024)	(0.3, 0.063, 0.03)	(0.223, 0.043, 0.021)	(0.229, 0.043, 0.021)	(0.283, 0.063, 0.031)	(0.271, 0.055, 0.027)
200	(0.188, 0.04, 0.02)	(0.244, 0.053, 0.026)	(0.187, 0.035, 0.017)	(0.196, 0.038, 0.019)	(0.234, 0.053, 0.026)	(0.223, 0.046, 0.023)
250	(0.162, 0.035, 0.018)	(0.209, 0.047, 0.023)	(0.167, 0.033, 0.016)	(0.172, 0.034, 0.017)	(0.204, 0.047, 0.023)	(0.189, 0.041, 0.02)
			MKIW (α, β, λ) = MKIW (1.1, 0.8, 1.2)			
25	(0.249, 0.451, 1.514)	(0.377, 0.626, 1.883)	(0.226, 0.427, 0.932)	(0.32, 0.39, 1.475)	(0.457, 0.602, 2.488)	(0.314, 0.547, 1.466)
50	(0.171, 0.271, 0.705)	(0.293, 0.415, 1.308)	(0.15, 0.253, 0.512)	(0.19, 0.258, 0.676)	(0.338, 0.411, 1.599)	(0.208, 0.332, 0.807)
100	(0.111, 0.179, 0.371)	(0.207, 0.297, 0.795)	(0.102, 0.17, 0.302)	(0.119, 0.18, 0.35)	(0.234, 0.301, 0.936)	(0.14, 0.227, 0.478)
150	(0.085, 0.136, 0.24)	(0.163, 0.239, 0.579)	(0.079, 0.131, 0.23)	(0.089, 0.141, 0.247)	(0.18, 0.243, 0.659)	(0.105, 0.174, 0.328)
200	(0.072, 0.124, 0.219)	(0.142, 0.206, 0.493)	(0.067, 0.115, 0.203)	(0.075, 0.126, 0.215)	(0.155, 0.211, 0.547)	(0.09, 0.151, 0.291)
250	(0.065, 0.104, 0.18)	(0.12, 0.184, 0.39)	(0.063, 0.104, 0.181)	(0.068, 0.112, 0.19)	(0.128, 0.185, 0.424)	(0.077, 0.128, 0.226)
			MKIW (α, β, λ) = MKIW (0.9, 0.3, 0.6)			
25	(0.569, 0.102, 0.247)	(0.651, 0.127, 0.476)	(0.638, 0.098, 0.278)	(0.503, 0.089, 0.272)	(0.589, 0.133, 0.627)	(0.598, 0.113, 0.345)
50	(0.358, 0.062, 0.15)	(0.416, 0.09, 0.304)	(0.414, 0.055, 0.159)	(0.361, 0.059, 0.161)	(0.409, 0.093, 0.361)	(0.373, 0.074, 0.201)
100	(0.255, 0.041, 0.105)	(0.277, 0.057, 0.165)	(0.269, 0.037, 0.099)	(0.244, 0.043, 0.103)	(0.272, 0.058, 0.181)	(0.258, 0.047, 0.123)
150	(0.212, 0.033, 0.082)	(0.23, 0.048, 0.128)	(0.208, 0.028, 0.077)	(0.191, 0.033, 0.081)	(0.227, 0.048, 0.136)	(0.212, 0.038, 0.097)
200	(0.175, 0.027, 0.068)	(0.195, 0.042, 0.107)	(0.177, 0.024, 0.064)	(0.162, 0.032, 0.075)	(0.195, 0.042, 0.112)	(0.18, 0.032, 0.081)
250	(0.152, 0.025, 0.062)	(0.167, 0.037, 0.093)	(0.161, 0.023, 0.061)	(0.156, 0.027, 0.066)	(0.167, 0.037, 0.096)	(0.156, 0.029, 0.071)
			MKIW (α, β, λ) = MKIW (0.3, 0.2, 0.5)			
25	(0.424, 0.065, 0.193)	(0.495, 0.08, 0.355)	(0.531, 0.052, 0.196)	(0.438, 0.051, 0.192)	(0.316, 0.081, 0.422)	(0.454, 0.073, 0.259)
50	(0.225, 0.039, 0.115)	(0.246, 0.051, 0.206)	(0.275, 0.034, 0.121)	(0.253, 0.037, 0.126)	(0.205, 0.052, 0.222)	(0.232, 0.042, 0.138)
100	(0.152, 0.025, 0.083)	(0.161, 0.035, 0.127)	(0.158, 0.023, 0.078)	(0.166, 0.026, 0.082)	(0.149, 0.035, 0.135)	(0.153, 0.028, 0.093)
150	(0.124, 0.021, 0.068)	(0.132, 0.03, 0.104)	(0.126, 0.018, 0.062)	(0.117, 0.02, 0.064)	(0.124, 0.03, 0.102)	(0.128, 0.024, 0.076)
200	(0.104, 0.018, 0.056)	(0.11, 0.025, 0.079)	(0.104, 0.015, 0.051)	(0.098, 0.019, 0.06)	(0.105, 0.025, 0.081)	(0.107, 0.02, 0.063)
250	(0.093, 0.016, 0.049)	(0.103, 0.023, 0.075)	(0.094, 0.014, 0.047)	(0.095, 0.017, 0.051)	(0.099, 0.022, 0.072)	(0.099, 0.018, 0.056)

Table 5 *Simulation results of MRE of the estimates for the MKIW distribution*

n	ADE (α, β, λ)	CVME (α, β, λ)	MLE (α, β, λ)	MPSE (α, β, λ)	OLSE (α, β, λ)	WLSE (α, β, λ)
			MKIW (α, β, λ) = MKIW (0.7, 0.5, 0.2)			
25	(1.003, 0.297, 0.325)	(1.631, 0.336, 0.522)	(1.209, 0.262, 0.330)	(1.458, 0.209, 0.341)	(1.113, 0.337, 0.616)	(1.293, 0.307, 0.424)
50	(0.591, 0.185, 0.214)	(0.841, 0.229, 0.807)	(0.645, 0.159, 0.192)	(0.846, 0.148, 0.207)	(0.713, 0.227, 0.308)	(0.730, 0.205, 0.247)
100	(0.388, 0.118, 0.142)	(0.495, 0.147, 0.177)	(0.393, 0.104, 0.126)	(0.431, 0.102, 0.134)	(0.459, 0.148, 0.182)	(0.449, 0.133, 0.157)
150	(0.321, 0.094, 0.119)	(0.429, 0.126, 0.151)	(0.319, 0.085, 0.103)	(0.327, 0.085, 0.104)	(0.404, 0.126, 0.154)	(0.387, 0.110, 0.133)
200	(0.269, 0.079, 0.101)	(0.349, 0.105, 0.131)	(0.268, 0.070, 0.086)	(0.280, 0.077, 0.095)	(0.334, 0.106, 0.132)	(0.319, 0.093, 0.112)
250	(0.232, 0.071, 0.089)	(0.298, 0.094, 0.114)	(0.239, 0.066, 0.080)	(0.246, 0.068, 0.082)	(0.291, 0.094, 0.115)	(0.270, 0.083, 0.098)
			MKIW (α, β, λ) = MKIW (1.1, 0.8, 1.2)			
25	(0.227, 0.563, 1.262)	(0.343, 0.782, 1.569)	(0.205, 0.534, 0.777)	(0.291, 0.487, 1.229)	(0.415, 0.753, 2.074)	(0.285, 0.684, 1.222)
50	(0.156, 0.339, 0.588)	(0.267, 0.519, 1.090)	(0.136, 0.316, 0.427)	(0.172, 0.322, 0.563)	(0.308, 0.514, 1.333)	(0.189, 0.415, 0.673)
100	(0.101, 0.224, 0.309)	(0.188, 0.371, 0.662)	(0.092, 0.212, 0.252)	(0.108, 0.225, 0.292)	(0.213, 0.377, 0.780)	(0.128, 0.284, 0.398)
150	(0.077, 0.170, 0.200)	(0.148, 0.299, 0.482)	(0.071, 0.164, 0.192)	(0.080, 0.176, 0.206)	(0.164, 0.303, 0.549)	(0.096, 0.218, 0.273)
200	(0.066, 0.155, 0.183)	(0.129, 0.257, 0.411)	(0.061, 0.143, 0.169)	(0.068, 0.157, 0.179)	(0.141, 0.263, 0.456)	(0.082, 0.189, 0.243)

MKIW (α, β, λ) = MKIW (0.7, 0.5, 0.2)						
n	ADE (α, β, λ)	CVME (α, β, λ)	MLE (α, β, λ)	MPSE (α, β, λ)	OLSE (α, β, λ)	WLSE (α, β, λ)
250	(0.059, 0.131, 0.150)	(0.109, 0.229, 0.325)	(0.057, 0.130, 0.151)	(0.062, 0.140, 0.158)	(0.117, 0.231, 0.353)	(0.070, 0.160, 0.189)
MKIW (α, β, λ) = MKIW (0.9, 0.3, 0.6)						
25	(0.632, 0.340, 0.411)	(0.723, 0.424, 0.793)	(0.708, 0.327, 0.463)	(0.559, 0.298, 0.453)	(0.654, 0.442, 1.045)	(0.664, 0.377, 0.575)
50	(0.398, 0.206, 0.250)	(0.462, 0.299, 0.507)	(0.460, 0.184, 0.265)	(0.401, 0.196, 0.268)	(0.454, 0.309, 0.602)	(0.415, 0.246, 0.335)
100	(0.283, 0.136, 0.174)	(0.308, 0.190, 0.275)	(0.299, 0.122, 0.166)	(0.271, 0.142, 0.171)	(0.302, 0.194, 0.301)	(0.287, 0.157, 0.204)
150	(0.236, 0.110, 0.137)	(0.255, 0.159, 0.214)	(0.231, 0.095, 0.129)	(0.212, 0.110, 0.135)	(0.252, 0.160, 0.226)	(0.235, 0.128, 0.162)
200	(0.194, 0.089, 0.113)	(0.216, 0.139, 0.178)	(0.197, 0.078, 0.106)	(0.180, 0.105, 0.125)	(0.217, 0.140, 0.186)	(0.200, 0.107, 0.135)
250	(0.169, 0.083, 0.103)	(0.186, 0.124, 0.156)	(0.178, 0.076, 0.101)	(0.173, 0.091, 0.109)	(0.186, 0.125, 0.160)	(0.173, 0.097, 0.118)
MKIW (α, β, λ) = MKIW (0.3, 0.2, 0.5)						
25	(1.413, 0.323, 0.386)	(1.649, 0.401, 0.709)	(1.771, 0.260, 0.393)	(1.459, 0.257, 0.384)	(1.053, 0.406, 0.845)	(1.514, 0.363, 0.517)
50	(0.749, 0.193, 0.230)	(0.821, 0.256, 0.412)	(0.917, 0.170, 0.243)	(0.842, 0.184, 0.253)	(0.682, 0.259, 0.445)	(0.773, 0.211, 0.277)
100	(0.507, 0.125, 0.165)	(0.537, 0.173, 0.254)	(0.525, 0.113, 0.156)	(0.554, 0.130, 0.165)	(0.496, 0.175, 0.269)	(0.511, 0.139, 0.186)
150	(0.412, 0.106, 0.135)	(0.439, 0.150, 0.208)	(0.421, 0.090, 0.123)	(0.391, 0.100, 0.127)	(0.415, 0.150, 0.204)	(0.425, 0.120, 0.152)
200	(0.345, 0.088, 0.113)	(0.365, 0.125, 0.158)	(0.347, 0.075, 0.102)	(0.327, 0.097, 0.120)	(0.350, 0.125, 0.161)	(0.356, 0.102, 0.125)
250	(0.311, 0.079, 0.099)	(0.342, 0.112, 0.149)	(0.313, 0.071, 0.095)	(0.317, 0.083, 0.103)	(0.330, 0.112, 0.144)	(0.330, 0.090, 0.111)

Table 6 *Simulation results of SD of the estimates for the MKIW distribution*

MKIW (α, β, λ) = MKIW (0.7, 0.5, 0.2)						
n	ADE (α, β, λ)	CVME (α, β, λ)	MLE (α, β, λ)	MPSE (α, β, λ)	OLSE (α, β, λ)	WLSE (α, β, λ)
25	(1.485, 0.205, 0.090)	(2.880, 0.221, 0.204)	(1.846, 0.182, 0.095)	(1.848, 0.127, 0.085)	(1.767, 0.224, 0.240)	(1.721, 0.203, 0.130)
50	(0.629, 0.121, 0.057)	(1.049, 0.154, 3.274)	(0.752, 0.106, 0.052)	(0.984, 0.092, 0.051)	(0.840, 0.147, 0.085)	(0.882, 0.132, 0.066)
100	(0.369, 0.074, 0.037)	(0.510, 0.093, 0.046)	(0.386, 0.068, 0.031)	(0.430, 0.065, 0.033)	(0.461, 0.094, 0.047)	(0.455, 0.083, 0.040)
150	(0.296, 0.059, 0.030)	(0.431, 0.079, 0.038)	(0.295, 0.053, 0.025)	(0.299, 0.055, 0.027)	(0.403, 0.079, 0.038)	(0.378, 0.070, 0.033)
200	(0.253, 0.050, 0.026)	(0.327, 0.067, 0.033)	(0.255, 0.045, 0.022)	(0.260, 0.049, 0.024)	(0.311, 0.067, 0.034)	(0.297, 0.058, 0.029)
250	(0.213, 0.044, 0.022)	(0.285, 0.059, 0.029)	(0.223, 0.042, 0.019)	(0.217, 0.042, 0.021)	(0.274, 0.059, 0.029)	(0.253, 0.051, 0.025)
MKIW (α, β, λ) = MKIW (1.1, 0.8, 1.2)						
25	(0.369, 0.587, 3.520)	(0.462, 0.821, 2.729)	(0.314, 0.539, 2.260)	(0.418, 0.493, 3.059)	(0.460, 0.796, 2.984)	(0.399, 0.730, 2.312)
50	(0.252, 0.340, 1.840)	(0.389, 0.529, 2.098)	(0.211, 0.327, 1.170)	(0.261, 0.328, 1.489)	(0.401, 0.518, 2.301)	(0.290, 0.429, 1.445)
100	(0.164, 0.228, 0.842)	(0.288, 0.370, 1.361)	(0.136, 0.213, 0.471)	(0.160, 0.222, 0.681)	(0.305, 0.370, 1.504)	(0.199, 0.291, 0.852)
150	(0.106, 0.170, 0.321)	(0.232, 0.300, 1.008)	(0.101, 0.169, 0.315)	(0.114, 0.177, 0.371)	(0.246, 0.301, 1.111)	(0.140, 0.218, 0.487)
200	(0.093, 0.156, 0.293)	(0.204, 0.262, 0.857)	(0.088, 0.147, 0.284)	(0.096, 0.154, 0.297)	(0.215, 0.263, 0.928)	(0.125, 0.194, 0.455)
250	(0.083, 0.131, 0.239)	(0.173, 0.232, 0.654)	(0.080, 0.131, 0.245)	(0.085, 0.134, 0.255)	(0.182, 0.234, 0.711)	(0.101, 0.161, 0.318)
MKIW (α, β, λ) = MKIW (0.9, 0.3, 0.6)						
25	(0.896, 0.141, 0.437)	(1.004, 0.169, 0.810)	(0.984, 0.135, 0.589)	(0.675, 0.111, 0.433)	(0.714, 0.174, 0.931)	(0.937, 0.153, 0.606)
50	(0.492, 0.080, 0.209)	(0.566, 0.116, 0.530)	(0.554, 0.073, 0.279)	(0.470, 0.074, 0.221)	(0.482, 0.118, 0.600)	(0.512, 0.097, 0.312)
100	(0.343, 0.051, 0.133)	(0.365, 0.073, 0.262)	(0.344, 0.047, 0.134)	(0.325, 0.049, 0.132)	(0.338, 0.074, 0.287)	(0.340, 0.059, 0.166)
150	(0.270, 0.042, 0.104)	(0.292, 0.060, 0.178)	(0.266, 0.036, 0.100)	(0.234, 0.040, 0.107)	(0.277, 0.061, 0.190)	(0.269, 0.048, 0.125)
200	(0.221, 0.034, 0.087)	(0.243, 0.052, 0.140)	(0.229, 0.030, 0.083)	(0.204, 0.036, 0.097)	(0.234, 0.052, 0.146)	(0.227, 0.041, 0.103)
250	(0.199, 0.031, 0.078)	(0.218, 0.047, 0.123)	(0.207, 0.028, 0.078)	(0.195, 0.031, 0.083)	(0.211, 0.047, 0.127)	(0.204, 0.036, 0.090)
MKIW (α, β, λ) = MKIW (0.3, 0.2, 0.5)						
25	(0.893, 0.089, 0.305)	(1.284, 0.107, 0.548)	(1.292, 0.072, 0.351)	(0.673, 0.062, 0.276)	(0.666, 0.105, 0.593)	(1.043, 0.099, 0.414)
50	(0.354, 0.050, 0.151)	(0.389, 0.066, 0.341)	(0.439, 0.044, 0.180)	(0.356, 0.047, 0.168)	(0.290, 0.066, 0.342)	(0.368, 0.055, 0.193)
100	(0.217, 0.032, 0.108)	(0.239, 0.044, 0.194)	(0.217, 0.029, 0.104)	(0.236, 0.030, 0.105)	(0.207, 0.044, 0.196)	(0.222, 0.035, 0.124)
150	(0.167, 0.026, 0.085)	(0.181, 0.038, 0.196)	(0.173, 0.023, 0.079)	(0.148, 0.024, 0.083)	(0.164, 0.037, 0.140)	(0.174, 0.030, 0.097)
200	(0.141, 0.022, 0.070)	(0.149, 0.031, 0.102)	(0.142, 0.019, 0.066)	(0.127, 0.022, 0.076)	(0.139, 0.031, 0.105)	(0.145, 0.025, 0.079)
250	(0.125, 0.020, 0.062)	(0.140, 0.029, 0.137)	(0.130, 0.018, 0.060)	(0.121, 0.019, 0.065)	(0.132, 0.028, 0.091)	(0.132, 0.023, 0.070)

Table 7 *Simulation results of MSE of the estimates for the MKIW distribution*

n	ADE (α, β, λ)	CVME (α, β, λ)	MLE (α, β, λ)	MPSE (α, β, λ)	OLSE (α, β, λ)	WLSE (α, β, λ)
			MKIW (α, β, λ) = MKIW (0.7, 0.5, 0.2)			
25	(2.276, 0.048, 0.008)	(8.817, 0.051, 0.044)	(3.546, 0.035, 0.009)	(4.099, 0.017, 0.009)	(3.182, 0.052, 0.065)	(3.203, 0.044, 0.018)
50	(0.401, 0.016, 0.003)	(1.167, 0.024, 10.723)	(0.576, 0.012, 0.003)	(1.147, 0.009, 0.003)	(0.716, 0.022, 0.008)	(0.807, 0.018, 0.005)
100	(0.137, 0.006, 0.001)	(0.267, 0.009, 0.002)	(0.149, 0.005, 0.001)	(0.211, 0.004, 0.001)	(0.213, 0.009, 0.002)	(0.209, 0.007, 0.002)
150	(0.089, 0.004, 0.001)	(0.194, 0.006, 0.002)	(0.088, 0.003, 0.001)	(0.098, 0.003, 0.001)	(0.164, 0.006, 0.002)	(0.147, 0.005, 0.001)
200	(0.065, 0.003, 0.001)	(0.111, 0.005, 0.001)	(0.065, 0.002, 0.001)	(0.074, 0.002, 0.001)	(0.098, 0.005, 0.001)	(0.090, 0.003, 0.001)
250	(0.046, 0.002, 0.001)	(0.082, 0.004, 0.001)	(0.050, 0.002, 0.000)	(0.050, 0.002, 0.000)	(0.075, 0.004, 0.001)	(0.064, 0.003, 0.001)
			MKIW (α, β, λ) = MKIW (1.1, 0.8, 1.2)			
25	(0.142, 0.385, 13.388)	(0.245, 0.717, 9.179)	(0.100, 0.344, 5.278)	(0.181, 0.247, 10.505)	(0.339, 0.633, 13.154)	(0.177, 0.573, 6.256)
50	(0.064, 0.125, 3.489)	(0.169, 0.283, 5.169)	(0.045, 0.115, 1.396)	(0.069, 0.109, 2.347)	(0.217, 0.269, 6.873)	(0.088, 0.192, 2.258)
100	(0.027, 0.054, 0.721)	(0.092, 0.137, 2.069)	(0.019, 0.047, 0.224)	(0.026, 0.052, 0.468)	(0.115, 0.138, 2.694)	(0.041, 0.086, 0.763)
150	(0.011, 0.029, 0.105)	(0.058, 0.090, 1.106)	(0.010, 0.030, 0.100)	(0.013, 0.034, 0.138)	(0.072, 0.091, 1.409)	(0.020, 0.048, 0.247)
200	(0.009, 0.025, 0.087)	(0.045, 0.069, 0.801)	(0.008, 0.022, 0.081)	(0.009, 0.026, 0.088)	(0.054, 0.071, 0.97 6)	(0.016, 0.038, 0.214)
250	(0.007, 0.018, 0.058)	(0.031, 0.054, 0.452)	(0.006, 0.018, 0.060)	(0.007, 0.020, 0.066)	(0.037, 0.055, 0.551)	(0.010, 0.026, 0.102)
			MKIW (α, β, λ) = MKIW (0.9, 0.3, 0.6)			
25	(0.813, 0.022, 0.194)	(1.013, 0.029, 0.733)	(0.983, 0.019, 0.354)	(0.480, 0.013, 0.205)	(0.575, 0.030, 1.092)	(0.885, 0.025, 0.393)
50	(0.244, 0.007, 0.044)	(0.320, 0.014, 0.305)	(0.311, 0.005, 0.079)	(0.241, 0.006, 0.051)	(0.259, 0.014, 0.416)	(0.263, 0.010, 0.102)
100	(0.119, 0.003, 0.018)	(0.134, 0.005, 0.072)	(0.118, 0.002, 0.018)	(0.113, 0.003, 0.017)	(0.118, 0.006, 0.090)	(0.117, 0.004, 0.028)
150	(0.073, 0.002, 0.011)	(0.085, 0.004, 0.033)	(0.071, 0.001, 0.010)	(0.058, 0.002, 0.011)	(0.079, 0.004, 0.039)	(0.073, 0.002, 0.016)
200	(0.049, 0.001, 0.008)	(0.059, 0.003, 0.020)	(0.052, 0.001, 0.007)	(0.043, 0.002, 0.009)	(0.056, 0.003, 0.023)	(0.052, 0.002, 0.011)
250	(0.040, 0.001, 0.006)	(0.047, 0.002, 0.015)	(0.043, 0.001, 0.006)	(0.040, 0.001, 0.007)	(0.046, 0.002, 0.017)	(0.042, 0.001, 0.008)
			MKIW (α, β, λ) = MKIW (0.3, 0.2, 0.5)			
25	(0.849, 0.009, 0.095)	(1.716, 0.012, 0.343)	(1.769, 0.005, 0.126)	(0.565, 0.004, 0.082)	(0.444, 0.011, 0.448)	(1.147, 0.010, 0.185)
50	(0.131, 0.003, 0.023)	(0.157, 0.004, 0.125)	(0.205, 0.002, 0.033)	(0.157, 0.002, 0.029)	(0.085, 0.004, 0.133)	(0.142, 0.003, 0.038)
100	(0.048, 0.001, 0.012)	(0.058, 0.002, 0.040)	(0.048, 0.001, 0.011)	(0.066, 0.001, 0.011)	(0.043, 0.002, 0.043)	(0.050, 0.001, 0.016)
150	(0.028, 0.001, 0.007)	(0.033, 0.001, 0.039)	(0.030, 0.001, 0.006)	(0.025, 0.001, 0.007)	(0.027, 0.001, 0.021)	(0.031, 0.001, 0.010)
200	(0.020, 0.001, 0.005)	(0.023, 0.001, 0.011)	(0.020, 0.000, 0.004)	(0.017, 0.001, 0.006)	(0.019, 0.001, 0.011)	(0.021, 0.001, 0.006)
250	(0.016, 0.000, 0.004)	(0.020, 0.001, 0.019)	(0.017, 0.000, 0.004)	(0.016, 0.000, 0.004)	(0.017, 0.001, 0.009)	(0.018, 0.001, 0.005)

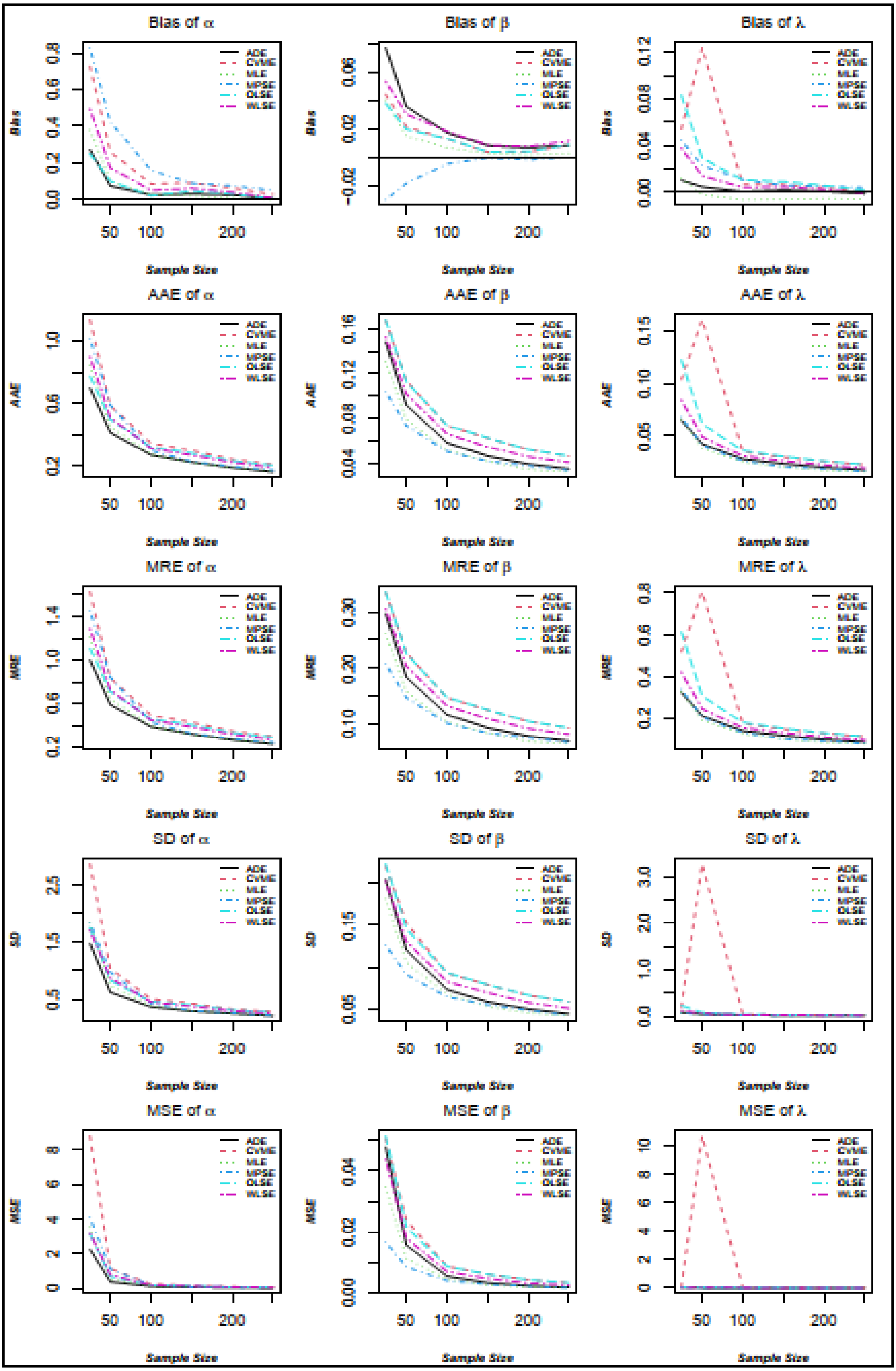

Fig. 3 *Graphical representation of the simulation outcomes for the bias, AAE, MRE, SD, and MSE for* $MKIW(\mathbf{0.7}, \mathbf{0.5}, \mathbf{0.2})$

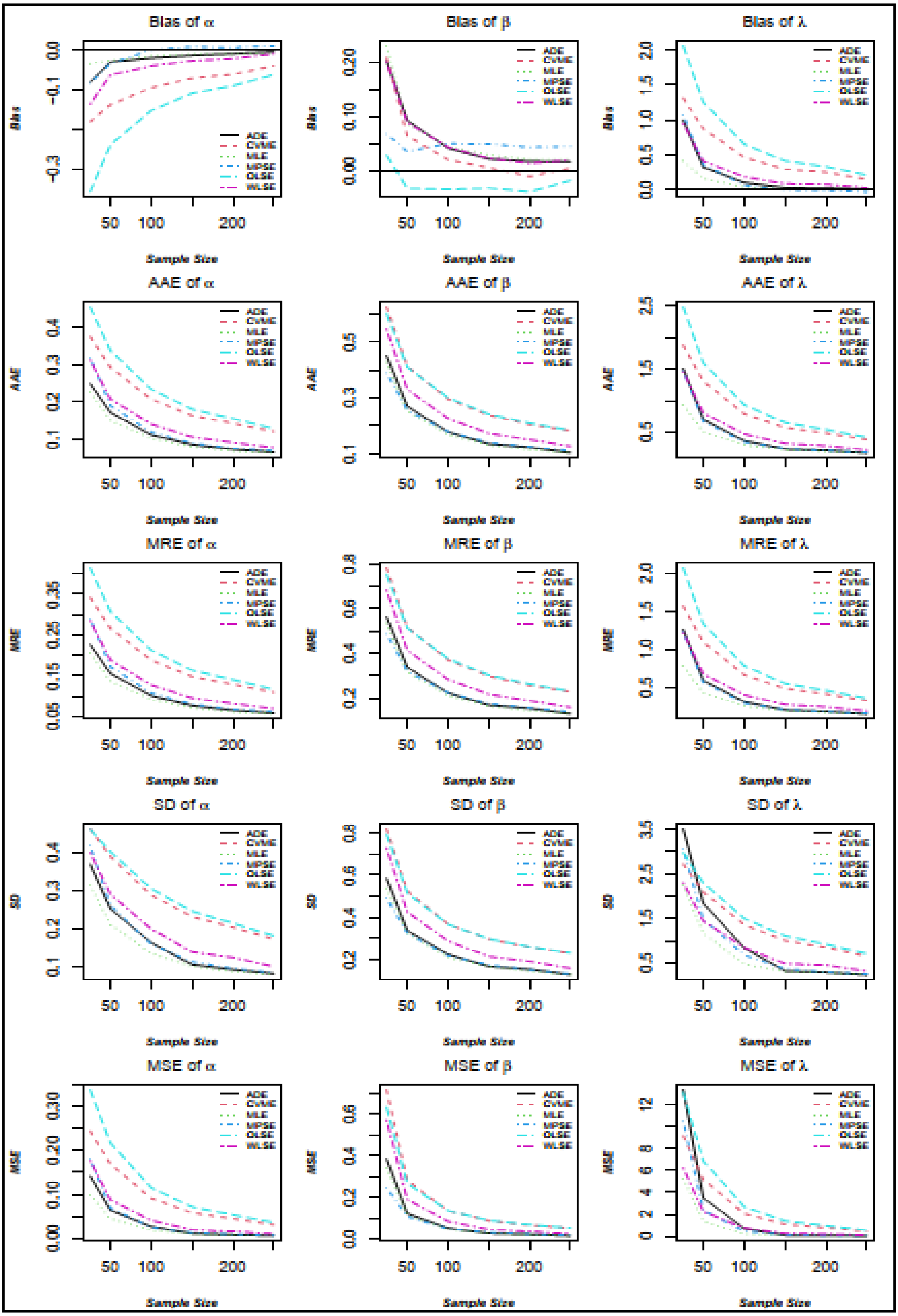

Fig. 4 *Graphical representation of the simulation outcomes for the bias, AAE, MRE, SD, and MSE for* $MKIW(\mathbf{1.1}, \mathbf{0.8}, \mathbf{1.2})$

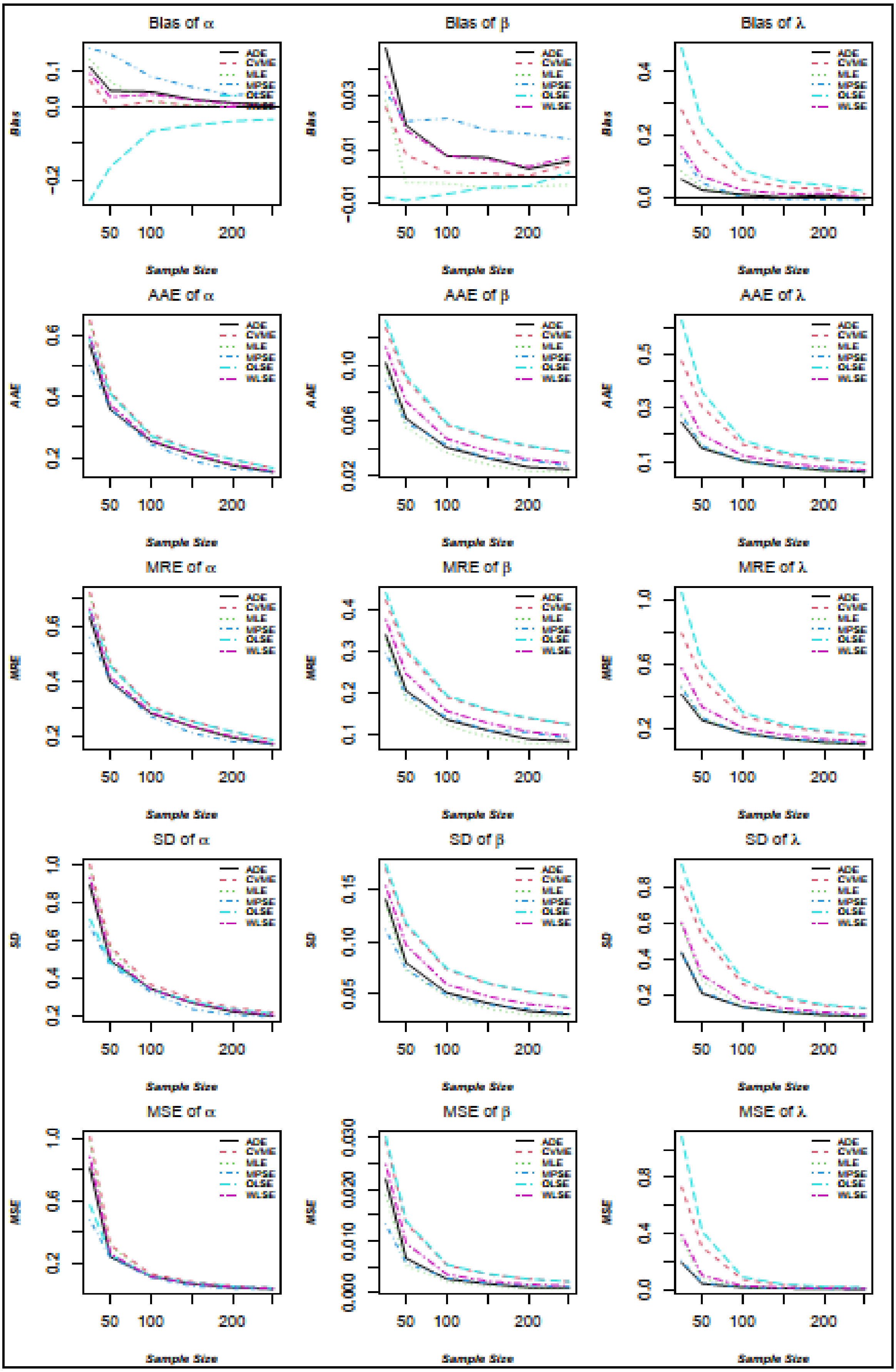

Fig. 5 *Graphical representation of the simulation outcomes for the bias, AAE, MRE, SD, and MSE for* $MKIW(\mathbf{0.9}, \mathbf{0.3}, \mathbf{0.6})$

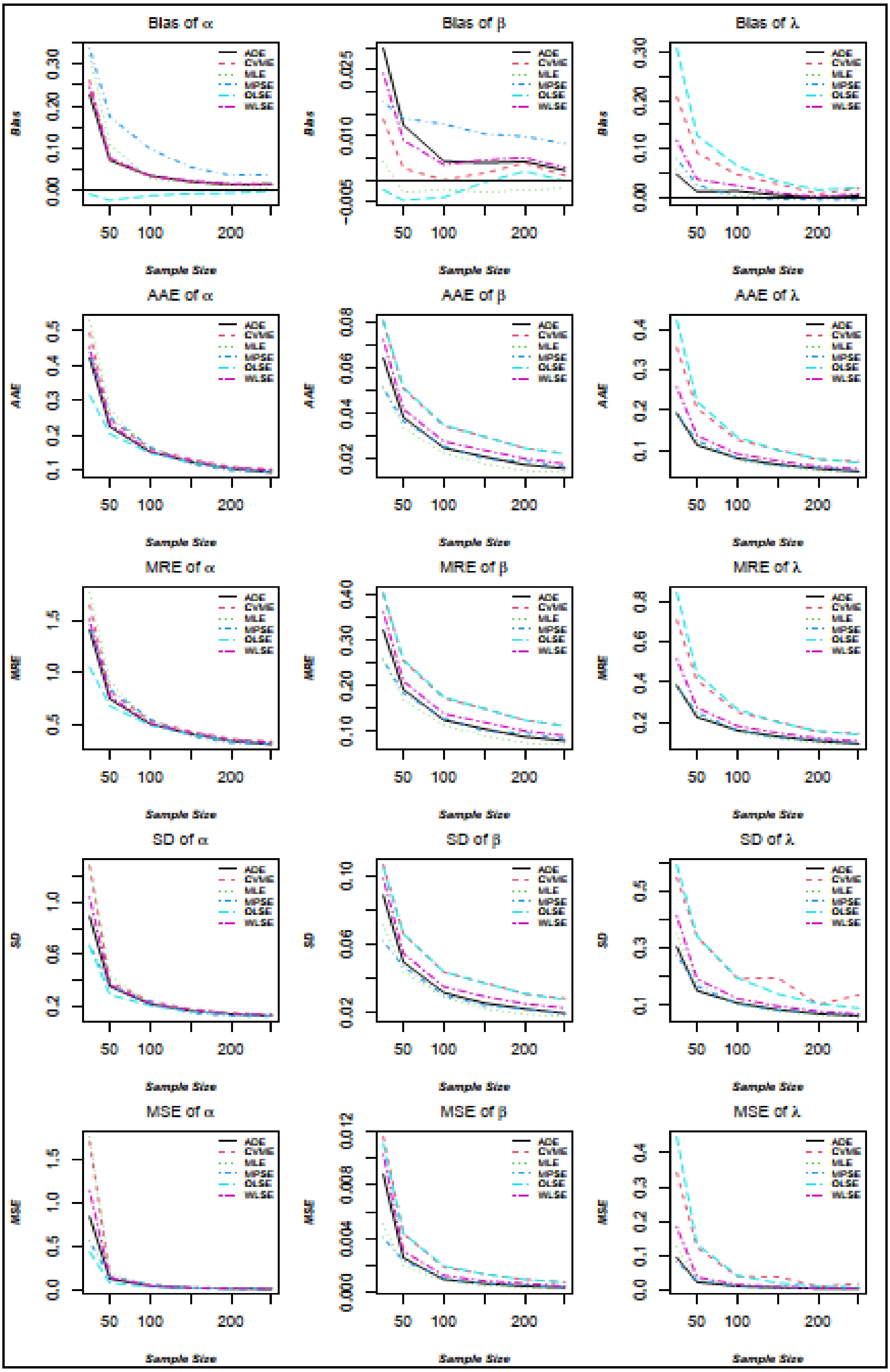

Fig. 6 *Graphical representation of the simulation outcomes for the bias, AAE, MRE, SD, and MSE for* $MKIW(\mathbf{0.3}, \mathbf{0.2}, \mathbf{0.5})$

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6. Real Data Applications

To assess the applicability and flexibility of the proposed MKIW model, four real-life datasets arising from different scientific fields were analysed. These datasets involve mortality rates, material strength measurements, medical survival times, and engineering failure times. The diversity of the datasets provides a comprehensive basis for evaluating the distributional performance under varying degrees of skewness and tail behaviour.

Model performance was assessed using multiple discrimination criteria specifically, the log-likelihood (ℓ), Akaike Information Criterion (AIC), Consistent AIC (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). Goodness-of-fit (GoF) was further evaluated using the Cramér-von Mises (W*), Anderson-Darling (A*), and Kolmogorov-Smirnov (KS) statistics, along with the corresponding KS *p*-value. The `AdequacyModel` package in R software was employed to compute all these statistical measures [36].
In general, superior model fit is indicated by lower values of these criteria and a higher KS *p*-value. The proposed MKIW model was compared with the IW distribution and some of its generalisations. The competing models and their respective PDFs are listed accordingly

- Exponentiated Generalised IW by [37]
$$f(x) = \lambda\theta\left(1 - e^{-(\alpha/x)^{\beta}}\right)^{\lambda-1}\left(1 - \left(1 - e^{-(\alpha/x)^{\beta}}\right)^{\lambda}\right)^{\theta-1}\left(\beta\alpha^{\beta}x^{-\beta-1}e^{-(\alpha/x)^{\beta}}\right), x, \alpha, \beta, \lambda, \theta > 0.$$

- Kumaraswamy IW by [26]
$$f(x) = \theta\lambda\left(\beta\alpha^{\beta}x^{-\beta-1}e^{-(\alpha/x)^{\beta}}\right)\left(e^{-(\alpha/x)^{\beta}}\right)^{\lambda-1}\left(1 - \left(e^{-(\alpha/x)^{\beta}}\right)^{\lambda}\right)^{\theta-1}, \qquad x, \alpha, \beta, \lambda, \theta > 0.$$

- Alpha Power IW by [38]
$$f(x) = \frac{ln(\lambda)}{\lambda - 1}\left(\beta\alpha^{\beta}x^{-\beta-1}e^{-(\alpha/x)^{\beta}}\right)\lambda^{e^{-(\alpha/x)^{\beta}}}, \quad x, \alpha, \beta, \lambda > 0.$$

- Topp-Leone IW by [23]
$$f(x) = 2\lambda\left(\beta\alpha^{\beta}x^{-\beta-1}e^{-(\alpha/x)^{\beta}}\right)\left(1 - e^{-(\alpha/x)^{\beta}}\right)\left(1 - \left(1 - e^{-(\alpha/x)^{\beta}}\right)^{2}\right)^{\lambda-1}, \qquad x, \alpha, \beta, \lambda > 0.$$

- IW by [2]
$$f(x) = \beta\alpha^{\beta}x^{-\beta-1}e^{-(\alpha/x)^{\beta}}, \qquad x, \alpha, \beta > 0.$$

6.1 Data I: COVID-19 Mortality Rates Dataset

The first dataset consists of 106 observations representing daily COVID-19 mortality rates recorded in Mexico over a period of 106 days, from March 4, 2020 to July 20, 2020. The mortality rates are expressed as proportions. This dataset was reported and analysed by [39] in their study on flexible exponent power-X distributions for modelling COVID-19 mortality data. The dataset exhibits notable variability and skewness, making it suitable for evaluating the fitting capability of lifetime and heavy-tailed distributions. The observations are: 8.826, 6.105, 9.391, 14.962, 10.383, 7.267, 13.220, 16.498, 11.665, 6.015, 10.855, 6.122, 6.656, 3.440, 5.854, 10.685, 10.035, 5.242, 4.344, 5.143, 7.630, 14.604, 7.903, 6.370, 3.537, 6.327, 4.730, 3.215, 9.284, 12.878, 8.813, 10.043, 7.260, 5.985, 6.412, 3.395, 4.424, 9.935, 7.840, 9.550, 3.499, 3.751, 6.968, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.077, 3.778, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 7.486, 6.625, 6.140, 4.909, 4.661, 5.392, 12.042, 8.696, 1.815, 3.327, 5.406, 6.182, 1.041, 1.800, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 3.218, 2.926, 2.601, 2.065, 3.029, 2.058, 2.326, 2.506, 1.923

6.2 Data II: Carbon Fibre Tensile Strength Dataset

The second dataset contains 69 observations on the tensile strength of carbon fibres tested under tension at a gauge length of 20 mm. The tensile strengths are measured in gigapascals (GPa). This dataset has been widely used in reliability and materials science literature; and was reported and analysed by [40] in the study of the Topp-Leone Weibull distribution. Owing to its moderate skewness and engineering relevance, the dataset is appropriate for assessing the adequacy of continuous probability models in strength analysis. The data are: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

6.3 Data III: Chemotherapy Survival Times Dataset

The third dataset comprises 46 observations representing the survival times, measured in years, of patients who received chemotherapy treatment. The data were originally reported by [41] and later utilised by [42] in the analysis of the New Lomax-G family of distributions. Survival datasets of this form are frequently characterised by positive skewness and non-monotonic hazard structures, thereby providing an important benchmark for validating lifetime distributions. The observations are 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

6.4 Data IV: Lifting Engine Failure Times Dataset

The fourth dataset reports the failure times of lifting engines used in giant industrial machines. The dataset consists of 39 observed failure times, measured in thousands of hours. This dataset was reported and analysed by [43] for reliability modelling and failure prediction. The observations are measured in time units as reported in the original study and given as: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0. This dataset is commonly employed in reliability engineering to evaluate the suitability of lifetime distributions for modelling mechanical component failures.

Table 8 presents the descriptive statistics of the datasets, supported by the box plots in Figure 7. It can be noted Data I and Data IV showing greater dispersion than Data II and Data III. Data I and Data III exhibit mild positive skewness, with Data I containing outliers suggesting a longer right tail, while Data II and Data IV show slight left skewness. The MKIW model effectively captures these distributional features, as illustrated by the PDF plots in Figure 1.

Table 8 *Descriptive statistics of the datasets*

Dataset	n	Mean	Median	Mode	Min	Max	Std Dev	Skewness	Kurtosis
Data I	106	5.8223	5.2795	8.8260	1.0410	16.4980	3.2499	0.9732	3.6661
Data II	69	2.4513	2.4780	2.3010	1.3120	3.5850	0.4951	-0.0282	2.9407
Data III	46	1.3247	0.7685	0.0470	0.0470	4.0330	1.2379	1.0061	2.7347
Data IV	39	6.2103	6.5000	7.3000	1.6000	9.0000	1.9625	-0.6233	2.6032

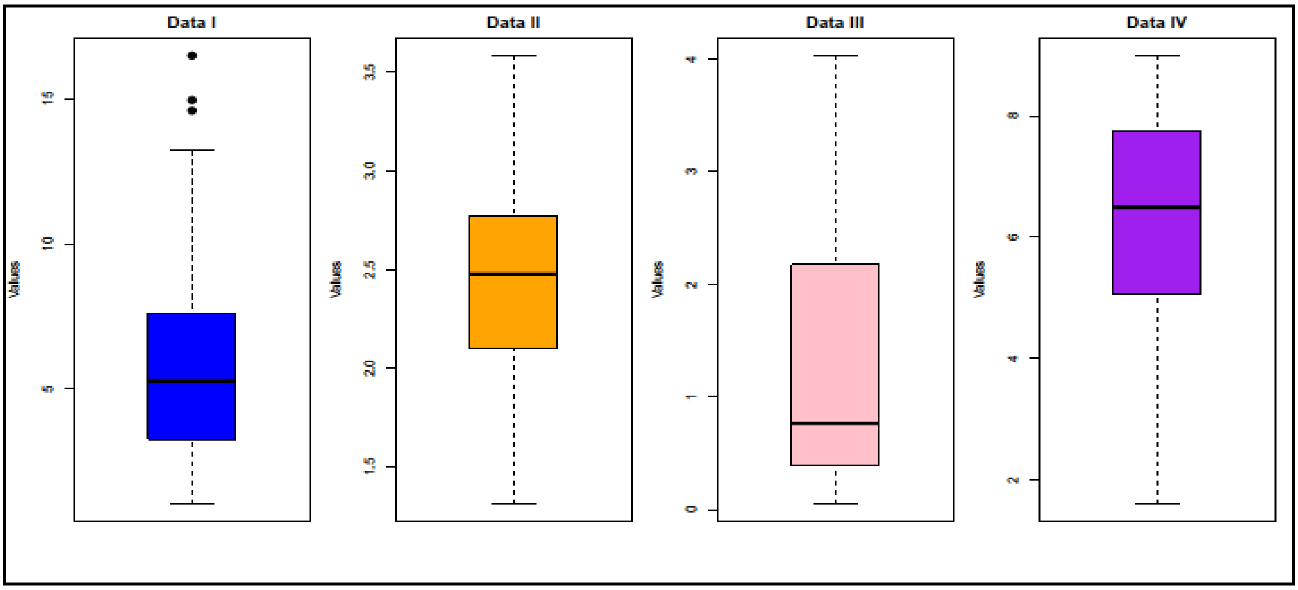

Fig. 7 *Box plots of the data*

Tables 9 to 12 present the MLEs and their corresponding standard errors (SEs), together with ℓ, AIC, BIC, CAIC, HQIC, W*, A*, KS statistics, and their associated p-values (KSpv) for each dataset. The results consistently indicate the adequate fit of the MKIW model, as it yields the lowest values for all discrimination and GoF criteria and the highest KS p-values across the datasets. Furthermore, the graphical assessments of model fit, including density estimations, empirical and fitted CDFs, probability-probability (PP), and quantile-quantile (QQ) plots, displayed in Figures 8 to 11, provide additional evidence supporting the favourable performance of the MKIW distribution compared with the competing models.

Table 9 *The MLEs, their SEs, discrimination criteria and GoF measures for data I*

Statistic	MKIW	EGIW	KIW	APIW	TLIW	IW
$\hat{\alpha}$	4.3458	18.8630	5.4388	2.4297	8.7190	3.7038
$\hat{\beta}$	1.0241	1.0344	0.6921	2.0332	1.4004	1.6834
$\hat{\lambda}$	1.2645	8.0735	2.6134	16.4715	0.4010	-
$\hat{\theta}$	-	0.3499	10.9434	-	-	-
SE ($\hat{\alpha}$)	0.3891	7.2984	4.1153	0.2716	4.2223	0.2270
SE ($\hat{\beta}$)	0.2716	0.3259	0.1534	0.1433	0.1873	0.1163
SE ($\hat{\lambda}$)	0.3847	4.2964	1.5127	12.4500	0.4070	-
SE ($\hat{\theta}$)	-	0.4125	7.7461	-	-	-
$\hat{\ell}$	260.9671	262.2513	262.7300	268.7177	267.7302	273.6070
AIC	527.9342	532.5027	533.4600	543.4353	541.4603	551.2141
BIC	535.9246	543.1564	544.1137	551.4256	549.4506	556.5409
CAIC	528.1695	532.8987	533.8560	543.6706	541.6956	551.3306
HQIC	531.1728	536.8207	537.7780	546.6738	544.6988	553.3731
W*	0.0380	0.0580	0.0726	0.1822	0.1624	0.2735
A*	0.2157	0.3443	0.4196	1.1418	1.0353	1.7496
KS	0.0534	0.0730	0.0725	0.0824	0.0875	0.0924
KSpv	0.9233	0.6242	0.6331	0.4679	0.3915	0.3256

Table 10 *The MLEs, their SEs, discrimination criteria and GoF measures for data II*

Statistic	MKIW	EGIW	KIW	APIW	TLIW	IW
$\hat{\alpha}$	2.0756	3.4638	3.2815	1.7967	2.8439	2.1459
$\hat{\beta}$	1.5611	2.3313	1.9671	5.1181	3.4534	4.1274
$\hat{\lambda}$	2.4754	6.3818	1.4359	19.5524	0.5363	-
$\hat{\theta}$	-	0.8341	9.7601	-	-	-
SE ($\hat{\alpha}$)	0.3307	0.7177	1.3147	0.0733	1.0981	0.0667
SE ($\hat{\beta}$)	1.1033	0.4197	0.3989	0.4250	0.8064	0.3382
SE ($\hat{\lambda}$)	1.8444	3.9393	1.1766	13.3295	1.0839	-
SE ($\hat{\theta}$)	-	0.5788	6.4499	-	-	-
$\hat{\ell}$	48.8821	53.0109	52.0179	57.9745	58.0462	63.6243
AIC	103.7642	114.0218	112.0357	121.9489	122.0923	131.2485
BIC	110.4665	122.9582	120.9722	128.6513	128.7946	135.7168
CAIC	104.1334	114.6468	112.6607	122.3182	122.4615	131.4304
HQIC	106.4232	117.5672	115.5811	124.6080	124.7513	133.0212
W*	0.0205	0.1147	0.0931	0.2307	0.2397	0.3723
A*	0.1813	0.8005	0.6560	1.5340	1.6067	2.4195
KS	0.0437	0.1048	0.0794	0.1078	0.1133	0.1324
KSpv	0.9994	0.4348	0.7771	0.3986	0.3382	0.1781

For each dataset, Figures 8 - 11 present the fitted PDFs, fitted CDFs, PP plots, and QQ plots comparing the MKIW model against the baseline IW distribution and some of its generalisations. The fitted CDF plots (bottom-left panels) demonstrate that the MKIW model more accurately tracks the empirical cumulative distribution than the IW, particularly in the upper tail where survival probabilities are low. This improvement is visually confirmed by the PP plots (top-right): the MKIW points adhere more closely to the diagonal than the IW points, with deviations concentrated in the extreme tails. The QQ plots (bottom-right) further validate this finding; the IW model shows systematic departures from the line of identity for extreme quantiles, whereas the MKIW points cluster tightly around the diagonal across the full range. This visual evidence indicates that the additional shape parameter λ enables the MKIW distribution to better accommodate the heavy right tails observed in Data I and Data III, as well as the slight left-tail departures in Data II and Data IV. The improved tail fit is not merely an artefact of model complexity; it reflects the genuine flexibility of the MKIW formulation in capturing diverse tail behaviour observed in all four datasets.

Table 11 *The MLEs, their SEs, discrimination criteria and GoF measures for data III*

Statistic	MKIW	EGIW	KIW	APIW	TLIW	IW
$\hat{\alpha}$	0.5891	9.6285	1.9423	0.2198	2.9085	0.4497
$\hat{\beta}$	0.4939	0.5627	0.3998	1.0556	0.7407	0.8758
$\hat{\lambda}$	1.4148	7.4771	1.6256	11.5608	0.3201	-
$\hat{\theta}$	-	0.3100	7.1578	-	-	-
SE ($\hat{\alpha}$)	0.1760	5.8380	1.4276	0.0695	4.5022	0.0806
SE ($\hat{\beta}$)	0.2102	0.1252	0.1012	0.1143	0.1548	0.0925
SE ($\hat{\lambda}$)	0.6906	4.2374	0.7508	12.5249	0.5414	-
SE ($\hat{\theta}$)	-	0.2271	5.3764	-	-	-
$\hat{\ell}$	58.1661	59.2666	60.0043	62.5239	61.8608	64.4154
AIC	122.3323	126.5332	128.0087	131.0479	129.7216	132.8307
BIC	127.8182	133.8478	135.3232	136.5338	135.2075	136.4880
CAIC	122.9037	127.5088	128.9843	131.6193	130.2930	133.1098
HQIC	124.3874	129.2733	130.7487	133.1029	131.7766	134.2008
W*	0.0517	0.0547	0.0647	0.1019	0.0924	0.1397
A*	0.3688	0.4148	0.4797	0.7333	0.6754	0.9660
KS	0.0772	0.0994	0.0981	0.1149	0.1062	0.1362
KS*pv*	0.9270	0.7167	0.7310	0.5401	0.6386	0.3301

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Table 12 *The MLEs, their SEs, discrimination criteria and GoF measures for data IV*

Statistic	MKIW	EGIW	KIW	APIW	TLIW	IW
$\hat{\alpha}$	2.1171	17.9880	2.4132	3.2474	8.4544	4.6276
$\hat{\beta}$	0.3078	1.2562	0.8661	2.5158	1.6834	1.9444
$\hat{\lambda}$	8.4131	14.4621	6.8891	18.0569	0.4986	-
$\hat{\theta}$	-	0.4126	17.0307	-	-	-
SE ($\hat{\alpha}$)	0.6950	4.3280	1.8011	0.3036	2.8372	0.4063
SE ($\hat{\beta}$)	0.0823	0.2538	0.1471	0.2587	0.2387	0.2065
SE ($\hat{\lambda}$)	2.4698	8.8007	4.4696	12.9916	0.4230	-
SE ($\hat{\theta}$)	-	0.2527	10.1917	-	-	-
$\hat{\ell}$	81.2390	87.5609	87.9211	94.6795	94.7201	98.8034
AIC	168.4781	183.1218	183.8423	195.3590	195.4402	201.6068
BIC	173.4687	189.7761	190.4965	200.3497	200.4308	204.9339
CAIC	169.1638	184.2983	185.0187	196.0447	196.1259	201.9401
HQIC	170.2687	185.5093	186.2298	197.1496	197.2308	202.8005
W*	0.0823	0.2550	0.2648	0.4585	0.4653	0.5758
A*	0.6166	1.6393	1.6919	2.7097	2.7527	3.3105
KS	0.0835	0.1708	0.1482	0.2042	0.1946	0.2399
KSpv	0.9487	0.2054	0.3582	0.0773	0.1042	0.0224

The ML estimates of the shape parameter λ varied substantially across the four datasets, reflecting the different underlying distributional characteristics. For Data I, the estimated λ was 1.2646, which exceeds 1 and indicates that the baseline hazard is amplified. This value is consistent with the observed positive skewness (0.9732) and the presence of outliers in the right tail; a $\lambda > 1$ increases the weight on the tail of the transformed variable, allowing the model to capture extreme mortality spikes. In Data II, λ was estimated at 2.4754, substantially larger than for Data I. This larger λ corresponds to a much lighter, near-symmetric distribution (skewness $= -0.0282$) and is required to tighten the density around the central values. Data III yielded $\lambda = 1.4148$, with the data exhibiting strong positive skewness (1.0061) and a long right tail. Here, $\lambda > 1$ again enhances the right-tail flexibility. Finally, Data IV gave an extremely high λ estimate of 8.4131. Despite the slight negative skewness (-0.6233) of this dataset, the very high λ value reflects the need for a very rapid decay in the upper tail of the distribution of the transformed variable, which in turn shapes the fitted density to match the observed clustering of failure times around the mode (6.5 - 7.3time units). In all four cases, the inclusion of λ leads to a statistically significant improvement over the baseline IW model, as confirmed by the likelihood ratio tests in Table 13.

Table 13 presents the likelihood ratio test (LRT) results comparing the proposed MKIW distribution with its baseline IW model. The hypotheses tested were:

$$H_0: \lambda = 1 \text{ against } H_1: \lambda \neq 1.$$

The LRT statistics were significant for all four datasets, with p-values less than 0.05. Hence, the null hypothesis was rejected in all cases, indicating that the additional shape parameter λ significantly improves the flexibility and fitting performance of the MKIW distribution relative to the classical IW model. These results confirm the superiority of the MKIW distribution for modelling the considered datasets.

Table 13 *LRT for the datasets*

Dataset	LogLik IW	LogLik MKIW	LRT Statistic	Deg. of Freedom	p-value
Data I	-273.6070	-260.9671	25.2798	1	4.96×10^{-7}
Data II	-63.6243	-48.8821	29.4843	1	5.64×10^{-8}
Data III	-64.4150	-58.1661	12.4985	1	4.07×10^{-4}
Data IV	-98.8034	-81.2390	35.1287	1	3.10×10^{-9}

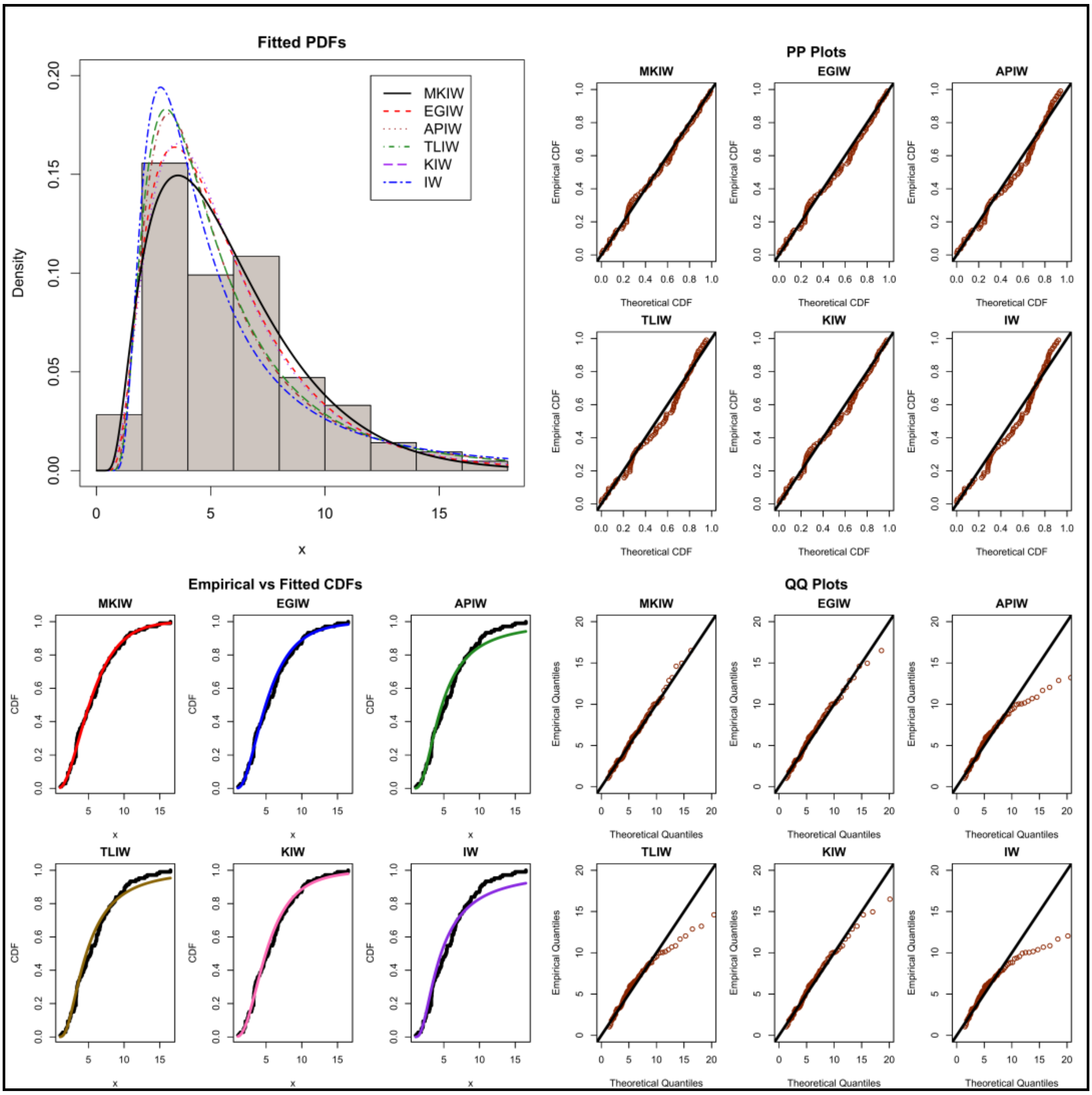

Fig. 8 *Fitted PDFs (top-left), PP (top-right), fitted CDFs (bottom-left) and QQ (bottom-right) plots of data I*

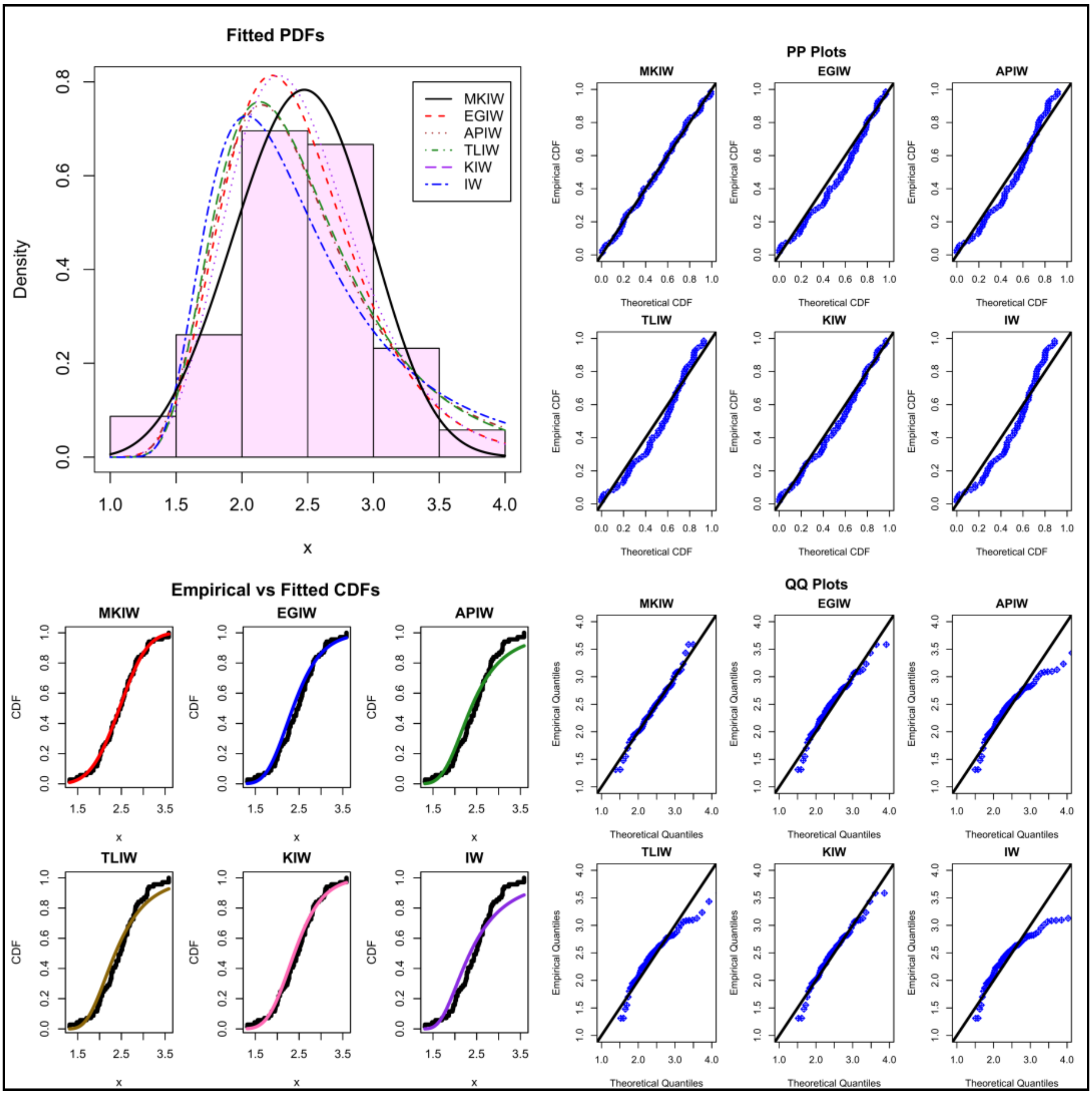

Fig. 9 *Fitted PDFs (top-left), PP (top-right), fitted CDFs (bottom-left) and QQ (bottom-right) plots of data II*

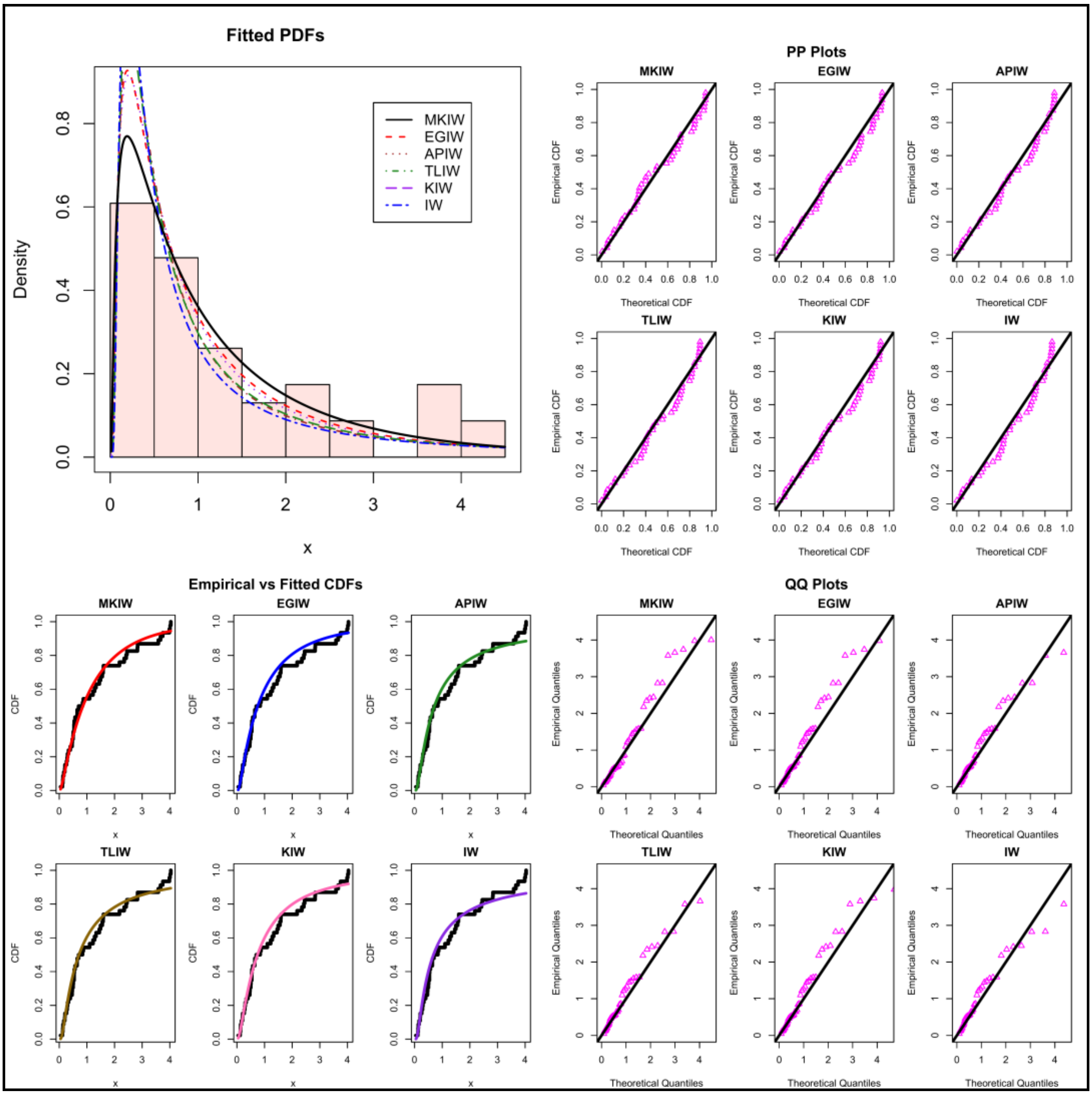

Fig. 10 *Fitted PDFs (top-left), PP (top-right), fitted CDFs (bottom-left) and QQ (bottom-right) plots of data III*

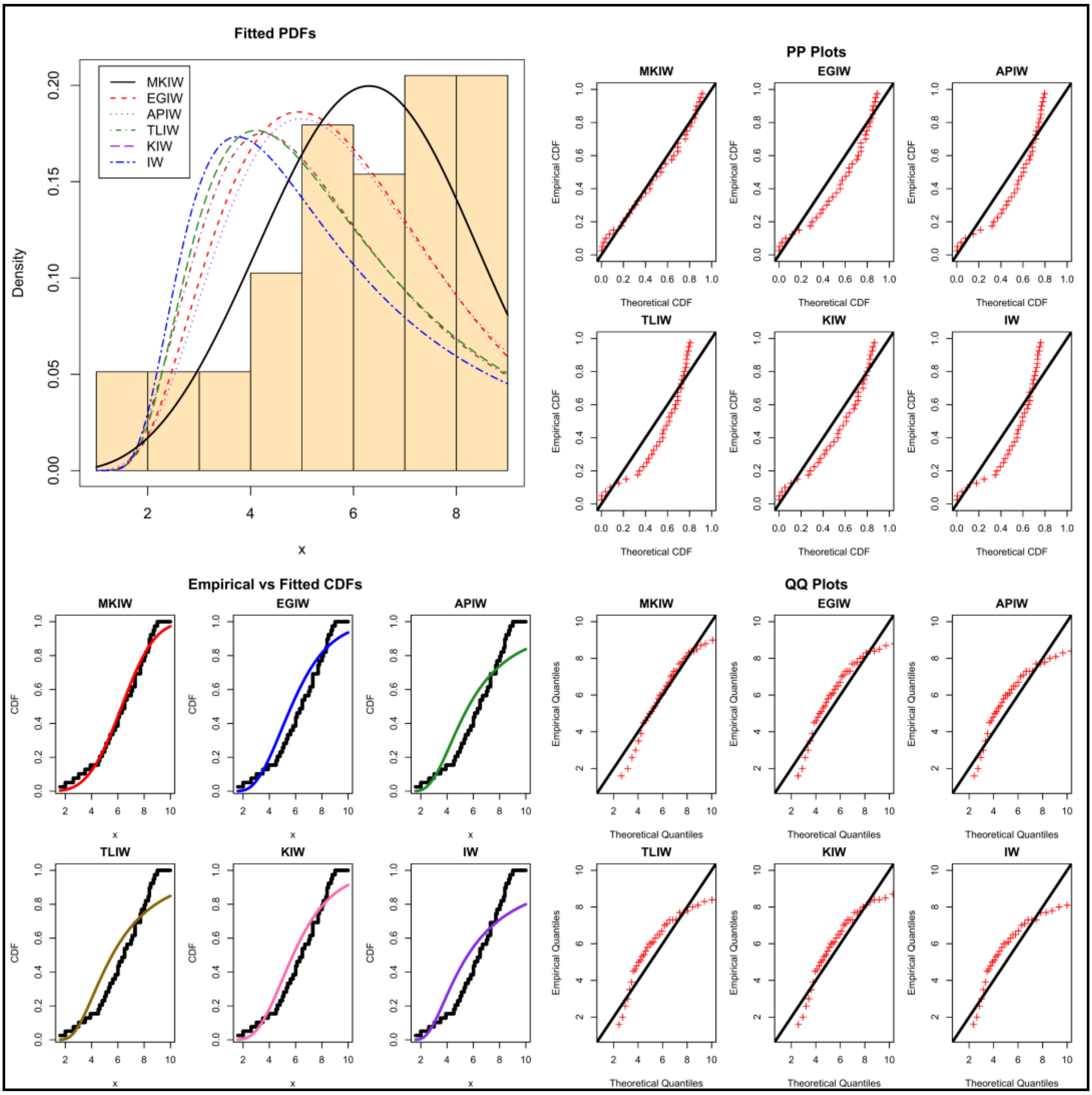

Fig. 11 *Fitted PDFs (top-left), PP (top-right), fitted CDFs (bottom-left) and QQ (bottom-right) plots of data IV*

7. Conclusion

This study proposed the Modified Kies Inverse Weibull (MKIW) distribution as a new extension of the Inverse Weibull distribution within the Modified Kies-G family framework. The proposed model introduces an additional shape parameter through an odds-based exponential generator, thereby enhancing flexibility in modelling skewness, tail behaviour, and hazard rate structures. Several important mathematical and statistical properties of the distribution were derived, including the quantile function, moments, skewness, kurtosis, median, mode, order statistics, and reliability measures. The MKIW distribution was shown to accommodate a wide range of density and hazard rate shapes, including decreasing, near-symmetric, and right-skewed densities, together with increasing, decreasing, and reversed bathtub hazard functions. These characteristics demonstrate the capacity of the proposed model to describe diverse lifetime phenomena more effectively than the classical Inverse Weibull (IW) distribution. Parameter estimation was investigated using six estimation techniques. The Monte Carlo simulation results revealed that the maximum product of spacings estimation method generally provides the most efficient estimators, closely followed by the maximum likelihood approach, based on average bias, average absolute bias, mean relative error, standard deviation, and mean squared error criteria. These findings confirm the reliability and stability of the proposed estimation procedures across different sample sizes and parameter configurations.

The practical usefulness of the MKIW distribution was demonstrated through applications to four real lifetime datasets. Comparative analyses showed that the proposed model consistently outperformed the classical IW distribution and several related extensions, including the exponentiated generalised IW, Kumaraswamy IW, Topp-Leone IW, and alpha power IW distributions. The superiority of the MKIW model was supported by lower values of information criteria such as AIC, BIC, CAIC, and HQIC, together with favourable goodness-of-fit statistics, including the Cramér-von Mises, Anderson-Darling, and Kolmogorov-Smirnov measures. The fitted densities, cumulative distribution functions, PP plots, and QQ plots further confirmed the adequacy of the proposed model. Finally, the MKIW distribution provides a flexible, analytically tractable, and practically useful alternative for modelling complex lifetime data. Its ability to accommodate diverse hazard rate behaviours and distributional shapes makes it a valuable addition to the existing family of Inverse Weibull generalisations.

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Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of the paper.

Author Contribution

The author confirms sole responsibility for the following: study conception and design, data collection, analysis and interpretation of results, and manuscript preparation.

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