JST

JOURNAL OF SCIENCE AND TECHNOLOGY
ISSN: 2229-8460 e-ISSN: 2600-7924

Vol. 18 No. 1 (2026) 119-128
https://publisher.uthm.edu.my/ojs/index.php/jst

Analytical Solution of the Advection-Diffusion Equation with Reaction and Spatially Dependent Coefficient for Exponentially Decay Source Injection

Nur Suhada Ibrahim[1], Zaiton Mat Isa[1,2]*

[1] *Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, Johor Bahru, 81310, Johor, MALAYSIA*

[2] *UTM-Centre for Industrial and Applied Mathematics (UTM-CIAM), Ibnu Sina Institute for Scientific and Industrial Research, Universiti Teknologi Malaysia, Johor Bahru, 81310, Johor, MALAYSIA*

*Corresponding Author: zaitonmi@utm.my
DOI: https://doi.org/10.30880/jst.2026.18.01.011

Article Info

Received: 14 November 2025
Accepted: 22 May 2026
Available online: 27 June 2026

Keywords

Advection diffusion equation, exponential decay, spatially dependent, laplace transform, analytic solution

Abstract

This study presents an analytical investigation of the one-dimensional advection-diffusion equation (ADE) involving reaction and source term with spatially dependent coefficient for exponential decay boundary conditions. The ADE is introduced and subsequently solved using the Laplace transformation approach. A change of spatial variables is first introduced to reduce the governing equation to one with constant coefficients. The Laplace transform is then applied, and the inverse transformation is evaluated using the Gaver–Stehfest algorithm. The results demonstrate that the temporal evolution of the solution is strongly governed by diffusion coefficient, inhomogeneity and the velocity indicating their critical roles in characterizing the transport dynamics. Higher diffusion coefficients produce broader concentration distributions. Similarly for higher flow velocity with particularly away from the source, the difference between the velocities becomes clearer. Meanwhile, medium inhomogeneity affects the spatial distribution of concentration as the pollutant moves further from the source, whereas higher flow velocity extends the transport distance and allows the contaminant to persist over a larger region.

1. Introduction

Diffusion is a fundamental process in which particles move from regions of higher concentration to areas of lower concentration, driven by concentration gradients. This process is central in fields such as environmental science, biology, and physics, as it governs the movement of solutes and contaminants through air, water, or soil [1]. Meanwhile advection refers to the transport of a substance by bulk motions of a fluid. Together, these processes are described by the advection-diffusion equation (ADE) a parabolic partial differential equation derived from the principle of conservation of mass using Fick's first law. According to Andallah and Khatun [2], the advection-diffusion equation is widely used to describe physical transport processes involving both diffusion and advection mechanisms, with applications in groundwater contamination, river pollution, and chemical transport modelling. In certain cases, the ADE is formulated with a reaction and source term.

In the context of the ADE, the type of coefficients plays a crucial role in determining the complexity and applicability of the model. For instance, studies by [3] and [4] used variable coefficients to account for spatial and temporal variations in the diffusion and advection coefficients. Previous studies have shown that ADE with variable coefficients helps model solute movement more realistically in finite and semi-infinite domains [5,6,7,8,9]. More recent works have further addressed spatially dependent transport coefficients and source effects to improve predictions in complex networks [10,11].

Research has also explored ADE with reaction terms to represent decay or biochemical processes. For example, Wadi et al. [12] modelled river pollution using a linear reaction term, while Samalerk and Pochai [13] applied a stable numerical approach with decay effects. Other works incorporated spatially dependent coefficients and first-order reactions to better simulate real environmental conditions [14,15]. More recent studies have continued to investigate advection diffusion reaction systems to improve the modelling of reactive contaminant transport under realistic environmental conditions. For instance, Owolabi and Alagoz [16] emphasized the importance of coupling diffusion, advection, and reaction mechanisms to better describe contaminant transport behaviour in porous media. Similarly, Coltman et al. [17] highlighted that spatial heterogeneity significantly affects dispersion behaviour and solute migration over larger spatial scales.

Boundary conditions are essential for defining how solutions to the ADE behave at the boundaries of a domain. Among the most commonly applied types are the Dirichlet condition, which specifies fixed values at the boundary, and the Neumann condition, which sets the derivative or flux at the boundary. Celis et al. [18] discussed these two common types specifically Dirichlet and Neumann boundary conditions. Dirichlet boundary conditions specify the value of the solution function directly at the boundary. On the other hand, Neumann boundary condition states that the derivative of the solution function such that a given value is imposed on the boundary of the domain. In physical applications, Dirichlet conditions are often used to model fixed concentration, whereas Neumann conditions are used to represent prescribed flux or concentration gradient across the boundary. These boundary conditions are widely applied in transport and diffusion models for solving partial differential equations. Exponentially decaying source represents a process in which the intensity or concentration of a source decreases continuously over time at an exponential rate. This is useful for physical situations where the solute concentration decreases naturally over time, such as pollutant removal, radioactive decay, or chemical reactions at the boundary layer.

Analytical solutions to the ADE are particularly valuable for understanding transport mechanisms and developing accurate predictive models. The analytical methods used for solving the ADE, such as Laplace or Fourier transforms, allow for handling time-dependent and spatially dependent parameters more precisely. Analytical solutions often serve as benchmarks for numerical solutions, providing exact solutions where computational methods might struggle with stability. The Laplace transform, in particular, has been effective in deriving solutions for ADEs with time-variable coefficients and exponential decay terms at the boundary [19]. Kumar [5] and Saleh et al. [20] provided analytical solutions using the Laplace transformation, while Ahsan [21] combined analytical and numerical approaches for river systems.

Previous studies provide a strong foundation for further investigation of ADE models involving variable coefficients and reaction processes. However, analytical solutions of one-dimensional ADE with spatially dependent coefficients, reaction and source term under exponentially decaying source injection remain unavailable, despite their relevance in modelling physical processes such as diminishing pollutant release, radioactive decay, and chemical degradation. These developments form the foundation for extending ADE analysis with exponential decay boundary conditions, as explored in this research. This present research addresses a gap in the current literature by focusing on a one-dimensional ADE with a spatially dependent coefficient, a source term, and an exponentially decaying boundary condition. The findings of this study may assist environmental engineers and relevant authorities in better understanding contaminant transport behavior, particularly in predicting pollutant dispersion. By providing a clearer picture of how contaminants spread over time and space, the model can support more accurate risk assessment and more informed decision-making in environmental monitoring, groundwater protection, and pollution control strategies.

2. Mathematical Formulation

The mathematical formulation is based on several simplifying assumptions where the transport process is treated as one-dimensional, and the medium is assumed to be continuous and homogeneous. Diffusion is described by Fick's law, while advection is represented as a constant flow. In addition, the reaction term is assumed to follow first-order kinetics, and the source term is prescribed. ADE with reaction and source term is written as,

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} - \mu C + \gamma \, , \tag{1}$$

where C is the concentration, D is the solute dispersion coefficient, V is the mean particle velocity, μ is a reaction and γ is the source term. A spatially dependent velocity is considered such that [8]

$$V(x,t) = u_0(1+ax) \,, \tag{2}$$

where u_0 is the initial velocity and a is the inhomogeneity. Solute dispersion is assumed proportional to the square of the velocity so it is considered as

$$D(x,t) = D_0(1+ax)^2 \,, \tag{3}$$

where D_0 is initial diffusion. Initially, there is C_0 concentration in the medium. Therefore

$$C(x,0) = C_0 \,. \tag{4}$$

At the boundary $x = 0$,

$$C(0,t) = C_1 e^{-mt} \,, \tag{5}$$

where C_1 is the initial concentration at the boundary and m is the decay rate constant. Meanwhile, at the other end $(x = L)$ it is assumed

$$\frac{\partial C}{\partial x}(L,t) = 0 \,, \tag{6}$$

where L is the length of the domain.

Although the problem may appear simpler than other more complex cases that have been solved, the combination of equations $(1) - (6)$ in obtaining an analytical solution of the ADE has not been reported previously, and thus represents a novel contribution.

3. Analytical Solution

To start, substitute (2) and (3) in (1) and the equation (1) will become

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D_0(1+ax)^2 \frac{\partial C}{\partial x} - u_0(1+ax)C\right) - \mu C + \gamma \,,$$

$$\frac{\partial C}{\partial t} = D_0(1+ax)^2 \frac{\partial^2 C}{\partial x^2} + 2aD_0(1+ax)\frac{\partial C}{\partial x} - u_0(1+ax)\frac{\partial C}{\partial x} - u_0 aC - \mu C + \gamma \,. \tag{7}$$

Introducing new variable X

$$X = \frac{1}{a(1+ax)} \,. \tag{8}$$

This substitution yields

$$\frac{\partial X}{\partial x} = -\frac{1}{(1+ax)^2} \,,$$
$$\frac{\partial C}{\partial x} = -\frac{1}{(1+ax)^2}\frac{\partial C}{\partial X} \,, \tag{9}$$

and

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{(1+ax)^4}\frac{\partial^2 C}{\partial X^2} \,. \tag{10}$$

By substituting (9) and (10) into (7) will end up with

$$\frac{\partial C}{\partial t} = D_0 \frac{1}{(1+ax)^2}\frac{\partial^2 C}{\partial X^2} - 2aD_0\left(\frac{1}{(1+ax)}\frac{\partial C}{\partial X}\right) + u_0\left(\frac{1}{(1+ax)}\frac{\partial C}{\partial X}\right) - u_0 aC - \mu C + \gamma \,, \tag{11}$$

and then substitute (8) into (11) to give

$$\frac{\partial C}{\partial t} = a^2X^2D_0\frac{\partial^2 C}{\partial X^2} - 2a^2XD_0\frac{\partial C}{\partial X} + aXu_0\frac{\partial C}{\partial X} - u_0aC - \mu C + \gamma \ . \quad (12)$$

Eq. (12) is further transformed into partial differential equation with constant coeeficient through this transformation

$$Z = \log aX = \log(1 + \ ax)\,. \quad (13)$$

This leads to

$$\frac{\partial C}{\partial t} = a^2D_0\frac{\partial^2 C}{\partial Z^2} + 2a^2D_0\frac{\partial C}{\partial Z} - au_0\frac{\partial C}{\partial Z} - u_0aC - \mu C + \gamma\,. \quad (14)$$

For the initial and boundary conditions Eq. (4) - (6), it will become

$$C(Z,0) = C_0 \ \ , \quad \text{for } 0 \le Z \le Z_0, Z_0 = \log(1 + aL) \ , \quad (15)$$
$$C(0,t) = C_1e^{-mt}\,, \quad (16)$$
$$\frac{\partial C}{\partial Z}(Z_0,t) = 0\,. \quad (17)$$

Eq. (14) can be written as

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial Z^2} - u\frac{\partial C}{\partial Z} - \alpha C + \gamma\,, \quad (18)$$

where $D = a^2D_0$, $u = au_0 - 2a^2D_0$ and $\alpha = au_0 + \mu$.
To solve Eq. (18), assume the solution in the form

$$C(Z,t) = F(Z,t)\exp\left[\frac{uZ}{2D} - \left(\frac{u^2}{4D} + \alpha\right)t\right] + \frac{\gamma}{\alpha} \ . \quad (19)$$

Differentiating with respect to t and z, and substituting into Eq. (18) reduces Eq. (18) to the diffusion equation

$$\frac{\partial F}{\partial t} = D\frac{\partial^2 F}{\partial Z^2} \ . \quad (20)$$

Following the assumption in Eq. (19), the initial and boundary condition. Eq. (15), (16), (17) become,

$$F(Z,0) = \left[C_0 - \frac{\gamma}{\alpha}\right]\exp\left[-\frac{uZ}{2D}\right], \quad (21)$$

$$F(0,t) = \left[C_1e^{-mt} - \frac{\gamma}{\alpha}\right]\exp\left[\left(\frac{u^2}{4D} + \alpha\right)t\right], \quad (22)$$

$$\frac{\partial F(Z_0,t)}{\partial Z} + F(Z_0,t)\frac{u}{2D} = 0\,. \quad (23)$$

By applying Laplace Transform into Eq. (20) along with initial condition Eq. (16) produces

J. of Science and Technology Vol. 18 No. 1 (2026) p. 119-128

$$\frac{\partial^2 \hat{F}}{\partial Z^2} - \frac{s}{D}\hat{F} = \frac{1}{D}\left[\frac{\gamma}{\alpha} - C_0\right] \exp\left[-\frac{uZ}{2D}\right] . \tag{24}$$

This is a second order nonhomogeneous equation, which by using the method of undetermined coefficient, results in

$$\hat{F}(Z,s) = C_2 \exp\left(-\sqrt{\frac{s}{D}}Z\right) + C_3 \exp\left(\sqrt{\frac{s}{D}}Z\right) - \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)\exp\left(\frac{uZ}{2D}\right)} . \tag{25}$$

It can be shown that the boundary conditions Eq. (22) and (23) in Laplace are

$$\hat{F}(0,t) = \frac{4C_1 D}{4sD + 4mD - u^2 - 4\alpha D} - \frac{\gamma}{\alpha}\frac{4D}{(4sD - u^2 - 4\alpha D)} , \tag{26}$$

and

$$\frac{\partial \hat{F}}{\partial Z}(Z_0,t) + \hat{F}(Z_0,t)\frac{u}{2D} = 0 . \tag{27}$$

By substituting Eq. (26) and (27) in Eq. (25), and after some necessary working, it is obtained that

$$C_2 = -C_3 + \frac{4C_1 D}{4sD + 4mD - u^2 - 4\alpha D} - \frac{\gamma}{\alpha}\frac{4D}{(4sD - u^2 - 4\alpha D)} + \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)} , \tag{28}$$

and

$$C_3 = \frac{\left(\sqrt{\frac{s}{D}} - \frac{u}{2D}\right)\left(\frac{4C_1 D}{4sD + 4mD - u^2 - 4\alpha D} - \frac{\gamma}{\alpha}\frac{4D}{(4sD - u^2 - 4\alpha D)} - + \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)}\right)}{\left(\sqrt{\frac{s}{D}} - \frac{u}{2D}\right) + \left(\sqrt{\frac{s}{D}} + \frac{u}{2D}\right) exp\left(2\sqrt{\frac{s}{D}}Z_0\right)} . \tag{29}$$

Finally,

$$\hat{F}(Z,s) = \left[\frac{4C_1 D}{4sD + 4mD - u^2 - 4\alpha D} \right. \tag{30}$$

$$- \frac{\left(\sqrt{\frac{s}{D}} - \frac{u}{2D}\right)\left(\frac{4C_1 D}{4sD + 4mD - u^2 - 4\alpha D} - \frac{\gamma}{\alpha}\frac{4D}{(4sD - u^2 - 4\alpha D)} + \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)}\right)}{\left(\sqrt{\frac{s}{D}} - \frac{u}{2D}\right) + \left(\sqrt{\frac{s}{D}} + \frac{u}{2D}\right)\exp\left(2\sqrt{\frac{s}{D}}Z_0\right)} + \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)}$$

$$\left. - \frac{\gamma}{\alpha}\frac{4D}{(4sD - u^2 - 4\alpha D)} \right] \exp\left(-\sqrt{\frac{s}{D}}Z\right)$$

$$+ \left[\frac{\left(\sqrt{\frac{s}{D}} - \frac{u}{2D}\right)\left(\frac{4C_1 D}{4sD + 4mD - u^2 - 4\alpha D} - \frac{\gamma}{\alpha}\frac{4D}{(4sD - u^2 - 4\alpha D)} + \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)}\right)}{\left(\sqrt{\frac{s}{D}} - \frac{u}{2D}\right) + \left(\sqrt{\frac{s}{D}} + \frac{u}{2D}\right)\exp\left(2\sqrt{\frac{s}{D}}Z_0\right)} \right] \exp\left(\sqrt{\frac{s}{D}}Z\right)$$

$$+ \frac{4D\left(C_0 - \frac{\gamma}{\alpha}\right)}{(u^2 - 4sD)\exp\left(\frac{uZ}{2D}\right)}.$$

The Gaver-Stehfest algorithm is utilized to obtain the inverse Laplace transform of Eq. (30), as the analytical inversion is expected to be highly complicated. This numerical inversion technique, along with the implementation of Eq. (30), is carried out using MATLAB software. The results are validated by ensuring that the obtained solution satisfies the governing equation and the imposed initial and boundary conditions. The Laplace transform method provides an exact analytical form in the transform domain, and the inverse solution is evaluated using the Gaver–Stehfest algorithm, which gives stable numerical results.

4. Result and Discussion

The results of the behavior of the solute concentration distribution in the domain at various parameter are illustrated graphically by using MATLAB. The fixed value of the reaction term, $\mu = 0.0006$, source term, $\gamma = 0.0011$, $C_0 = 1, C_1 = 10$ and $m = 0.5$ are used for all the concentrations.

Fig.1 presents the concentration profiles at different times, with fixed diffusion coefficient, $D_0 = 0.13$ m^2/day and velocity, $u_0 = 0.05$ m/day [22]. The inhomogeneity is also fixed at $a = 0.01m^{-1}$. At early times $t = 0.5$ day and $t = 1$ day, the concentration is highest at $x = 0$ and drops until $x = 1$, while after $x > 1$ the curve flattens. This indicates that during the initial stages, the pollutant remains more localized near the origin but begins to disperse rapidly with increasing distance. However, at later times $t = 2$ days and $t = 5$ days, at first there is a small increase in concentration before it begins to drop. For $t = 2$ days, the curve flattens around $x = 2$, while for $t = 5$ days, this flattening occurs around $x = 4$. This behavior is consistent with advection-diffusion theory under time-dependent transport processes, in which diffusion gradually reduces the concentration gradient and spreads the pollutant over a wider spatial region as time increases [23]. Compared to models with constant source or steady state assumptions, the present model provides a more realistic description by incorporating time dependent exponential decay at the source. Physically, this means that pollutants released into a medium initially remain localized near the source, but with increasing time they migrate further into the domain, resulting in a wider but less concentrated distribution. These results show that the proposed model is able to capture the expected temporal evolution of contaminant transport under exponentially decaying source conditions.

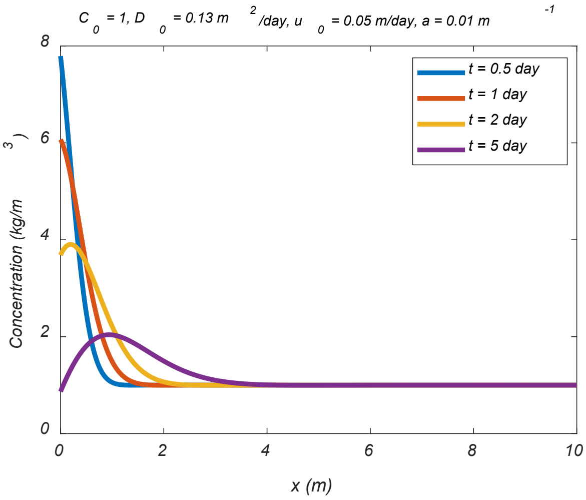
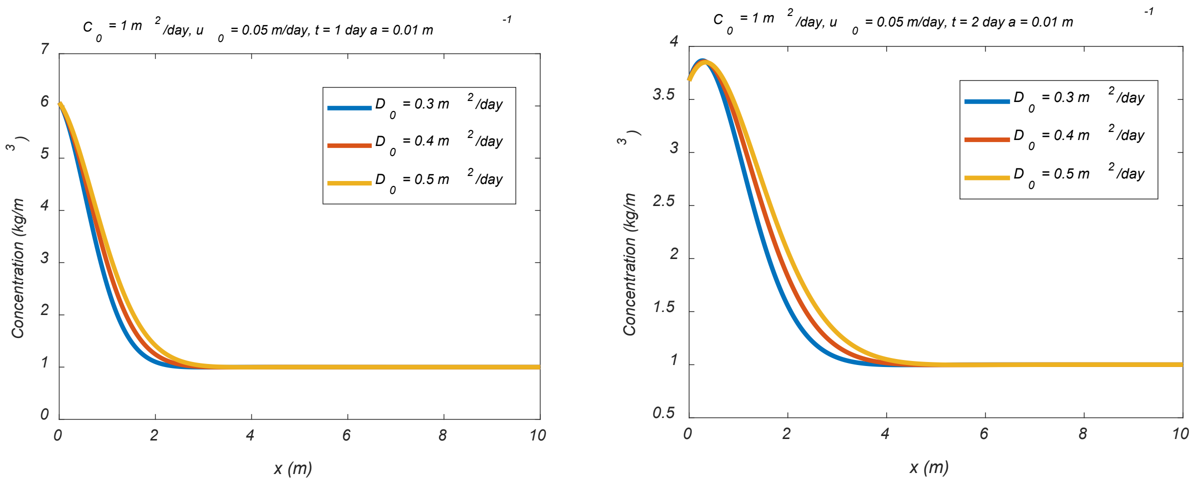


Fig. 1 *Concentration distribution based on parameters for various time,* ***t***

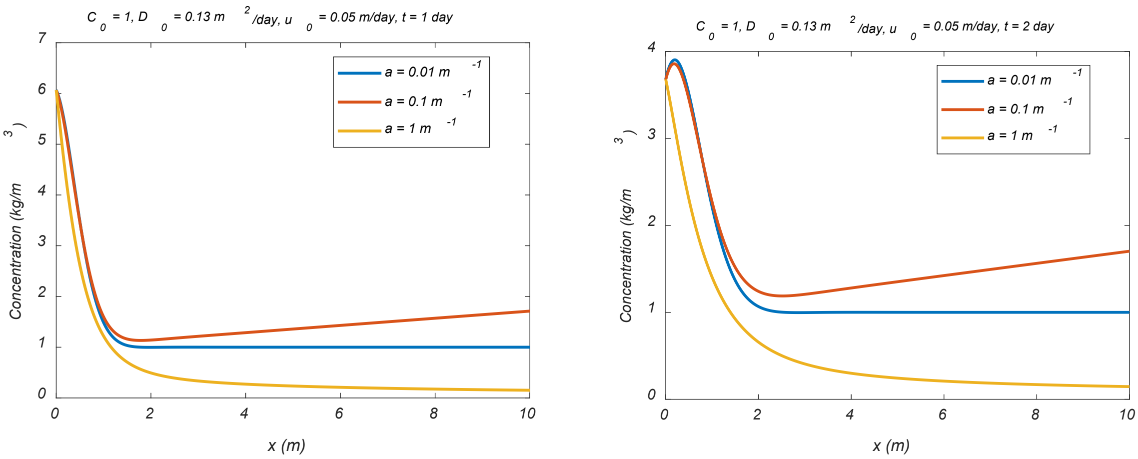
The results shown in Fig. 2 illustrate the concentration profiles for a fixed velocity $u_0 = 0.05$ m/day at times $t = 1$ and $t = 2$ days, under a varying diffusion coefficient D_0. The parameter D_0 represents the rate at which the pollutant spreads due to molecular diffusion. At $t = 1$ day, all curves peak at $x = 0$, indicating the highest concentration at the source. Then, the concentration drops and flattens around $x > 2$. Meanwhile for the graph at $t = 2$ days, all curves show a slight increase near the origin before dropping and flattening around $x > 3$. This reduction at the origin is physically justified by decline in source, when comparing the two, the peak is higher at $t = 1$ than at the later time $t = 2$. The lower concentration near the source at later time is attributed to the exponentially decaying boundary condition, where the source concentration decreases over time. In addition, higher values of D_0 produce broader concentration profiles, indicating faster pollutant spreading and dilution. This behavior is physically consistent with diffusion theory, where larger diffusion coefficients enhance solute dispersion over a wider spatial domain. From a physical perspective, a higher diffusion coefficient indicates greater molecular mixing, allowing the pollutant to spread more rapidly and reducing of high localized concentration buildup near the source.



a) $t = 1$ day   b) $t = 2$ day

Fig. 2 *Concentration distribution based on parameter obtained at $t = 1$ and 2 day for various diffusion, D_0*

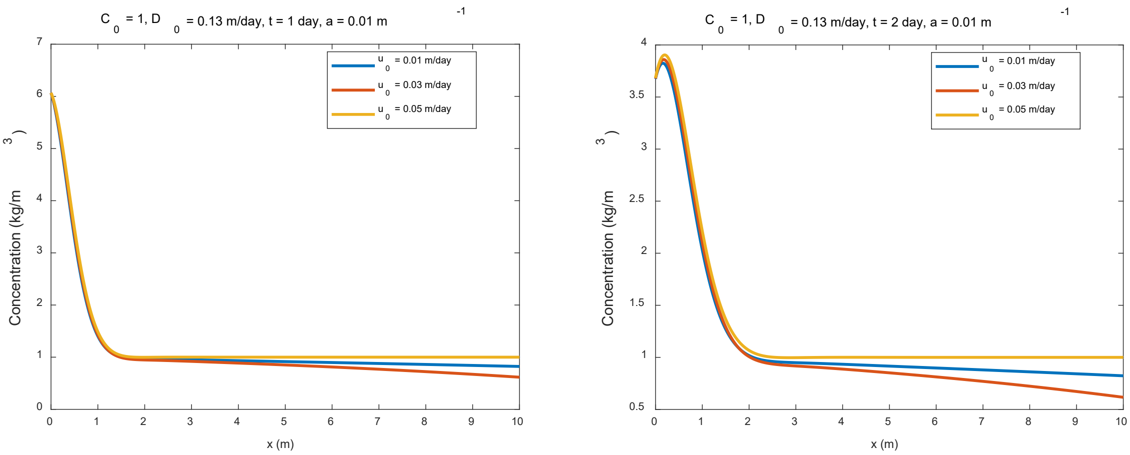
The graphs in Fig. 3 compare concentration profiles for the same diffusion coefficient D_0 and velocity u_0 but different medium inhomogeneity a. For both times, the peak concentration occurs at the source $x = 0$ with a noticeable decrease at $t = 2$ days, consistent with the exponential decay behavior at the boundary. The comparison also reveals that variations among different inhomogeneity parameters are not clearly distinguishable near the source. However, at larger distances, distinct differences become evident. This indicates that the influence of medium inhomogeneity becomes more pronounced further from the source, reflecting the systems sensitivity to spatial variations in transport properties. Physically, this suggests that variations in the medium properties, such as porosity or permeability, have a limited influence close to the source but increasingly affect pollutant migration as the solute travels through the domain. This behavior is consistent with recent studies by Cekmer et al. [24] on solute transport in heterogeneous porous media, where the influence of spatial variations in medium properties becomes more significant at larger distances from the source due to enhanced dispersive effects. These results highlight the importance of accounting for spatial variations in the medium, as they can significantly affect the final pollutant distribution even when the behaviors near the source are similar. Compared to standard models that assume a homogeneous medium, the present model provides a more realistic description of contaminant transport in spatially varying environments.



(a) $t = 1$ day **(b)** $t = 2$ day

Fig. 3 *Concentration distribution based on parameter obtained at* $\boldsymbol{t = 1}$ ***and*** $\boldsymbol{2}$ *day for various inhomogeneity,* $\boldsymbol{a}$

Lastly, similar behaviour is observed in Fig. 4 where the diffusion coefficient D_0 and inhomogeneity, a are kept fixed at two times while the velocity u_0 varies. At the boundary $x = 0$, all curves start with the same concentration and then drop sharply. Further away from the source, the difference between the velocities becomes clearer. This effect is more obvious at $t = 2$ days, where the higher velocity flow produces a longer concentration tail compared to the lower velocity. This indicates that velocity plays an important role in controlling how far the contaminant can travel and remain in the system. Physically, higher flow velocity allows the pollutant to move more rapidly and persist over a larger spatial region before being reduced by diffusion and reaction effects. These results show that velocity strongly influences both the shape of the concentration profile and the transport distance of contaminants, particularly over longer time periods. Compared to lower velocity conditions, higher velocity transport may increase the extent of pollutant migration, making flow velocity an important factor in contaminant transport analysis.



a) $t = 1$ day **(b)** $t = 2$ day

Fig. 4 *Concentration distribution based on parameter obtained at* $\boldsymbol{t = 1}$ ***and*** $\boldsymbol{2}$ *day for various velocity,* $\boldsymbol{u_0}$

5. Conclusion

This research derived analytical solutions for the one-dimensional ADE with spatially dependent, diffusion and velocity coefficients, including a reaction and source term, under exponential decay boundary conditions. Simulation results, carried out using MATLAB, highlight the influence of key physical parameters including diffusion, the inhomogeneity and flow velocity. Recognizing the critical influence of this parameter proves the importance of selecting accurate parameter values to predict real pollutant dispersion. Such precise predictions are vital for designing effective mitigation strategies to control and manage pollution in river systems.

Acknowledgement

The authors would also like to thank Faculty of Science, Universiti Teknologi Malaysia for providing the facilities and resources necessary to conduct this research.

Conflict of Interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

Author Contribution

The authors confirm contribution to the paper as follows: ***study conception and design:*** *Nur Suhada Ibrahim, Zaiton Mat Isa;* ***deriving analytic solution:*** *Nur Suhada Ibrahim;* ***analysis and interpretation of results:*** *Nur Suhada Ibrahim, Zaiton Mat Isa;* ***draft manuscript preparation:*** *Nur Suhada Ibrahim. All authors reviewed the results and approved the final version of the manuscript.*

References

[1] Biology Dictionary. (2019, October 4). Diffusion. https://biologydictionary.net/diffusion/

[2] Andallah, L., & Khatun. (2020). Numerical solution of advection-diffusion equation using finite difference schemes. Bangladesh Journal of Scientific and Industrial Research, 55(1), 15–22. https://doi.org/10.3329/bjsir.v55i1.46728

[3] Ayoub, A., Ghamkhar, M., Arif, A., & Moazzam, A. (2021). Laplace Transforms Techniques on Equation of Advection-Diffusion in One-Dimension with Semi-Infinite Medium to Find Analytical Solution. https://doi.org/10.33329/bomsr.9.3.86

[4] Kumar, A. K. A. (2012). Analytical Solutions of One-Dimensional Temporally Dependent Advection-Diffusion Equation along Longitudinal Semi-Infinite Homogeneous Porous Domain for Uniform Flow. *IOSR Journal of Mathematics*, *2*(1), 1–11. https://doi.org/10.9790/5728-0210111

[5] Kumar, A., Jaiswal, D. K., & Kumar, N. (2009). Analytical solutions of one-dimensional advection-diffusion equation with variable coefficients in a finite domain. *Journal of earth system science*, *118*, 539-549.

[6] Mojtabi, A., & Deville, M. O. (2014). One-dimensional linear advection–diffusion equation: Analytical and finite element solutions. *Computers & Fluids*, *107*, 189–195. https://doi.org/10.1016/j.compfluid.2014.11.006

[7] Ahmed, M., Zainab, Q. U. A., & Qamar, S. (2017). Analysis of One-Dimensional Advection–Diffusion Model with Variable Coefficients Describing Solute Transport in a Porous medium. *Transport in Porous Media*, *118*(3), 327–344. https://doi.org/10.1007/s11242-017-0833-0

[8] Rahaman, Muhammad & Takia, Humaira & Hasan, Md. Kamrul & Hossain, Bellal & Mia, Shamim & Hossen, Khokon. (2022). Application of Advection Diffusion Equation for Determination of Contaminants in Aqueous Solution: A Mathematical Analysis. Applied Mathematics and Physics. 10. 24-31

[9] Malik, S., Ejaz, S. T., Akgül, A., & Hassani, M. K. (2024). Exploring the advection-diffusion equation through the subdivision collocation method: a numerical study. Scientific Reports, 14(1). https://doi.org/10.1038/s41598024-52059-7

[10] Kim, A. S. (2020). Complete analytic solutions for convection-diffusion-reaction-source equations without using an inverse Laplace transform. *Scientific Reports*, *10*(1). https://doi.org/10.1038/s41598-020-63982-w8

[11] Shilsar, M. J. F., Mazaheri, M., & Samani, J. M. V. (2023). A semi-analytical solution for one-dimensional pollutant transport equation in different types of river networks. *Journal of Hydrology*, *619*, 129287. https://doi.org/10.1016/j.jhydrol.2023.129287

[12] Wadi, A. S., Dimian, M. F., & Ibrahim, F. N. (2014). Analytical solutions for one-dimensional advection–dispersion equation of the pollutant concentration. *Journal of Earth System Science*, *123*(6), 1317–1324. https://doi.org/10.1007/s12040-014-0468-2

[13] Samalerk, P., & Pochai, N. (2018). Numerical Simulation of a One-Dimensional Water-Quality Model in a Stream Using a Saulyev Technique with Quadratic Interpolated Initial-Boundary Conditions. *Abstract and Applied Analysis*, *2018*, 1–7. https://doi.org/10.1155/2018/1926519

[14] Ojwando, C. A., Onyango, T., Kimathi, M., & Obiero, B. (2020). Analytical Solution of a Non - Homogeneous One - Dimensional Advection Diffusion Equation With Temporally Varying Coefficients . *International Journal of Engineering Sciences & Research Technology*, *9*(12), 49–58. https://doi.org/10.29121/ijesrt.v9.i12.2020.7

[15] Abbasi, M., Madani, M., Sharifi, M., & Kazemi, A. (2021). Analytical solutions of advection-dispersion-reaction equation with first decay under constant and time-dependent boundary conditions: Mass transfer shape factor effects. *Groundwater for Sustainable Development*, *15*, 100691. https://doi.org/10.1016/j.gsd.2021.100691

[16] Owolabi, K. M., & Alagoz, S. (2025). Advection–diffusion–reaction modeling of contaminant transport in groundwater: Analysis and simulation. *Nonlinear Science.*, *5*, 100083. https://doi.org/10.1016/j.nls.2025.100083

[17] Coltman, E., Schneider, M., & Helmig, R. (2025). Data-Driven Closure Parametrizations with Metrics: Dispersive Transport. *Transport in Porous Media*, *152*(5). https://doi.org/10.1007/s11242-025-02168-2

[18] Celis, M., Wanderley, J., & Neves, M. (2017). Numerical simulation of dam breaking and the influence of sloshing on the transfer of water between compartments. Ocean Engineering, 146, 125–139. https://doi.org/10.1016/j.oceaneng.2017.09.029

[19] Essa, K. S. M., Etman, S. M., El-Otaify, M. S., Embaby, M., Mosallem, A. M., & Shalaby, A. S. (2021). Different solutions of the diffusion equation and its applications. Beni-Suef University Journal of Basic and Applied Sciences, 10(1). https://doi.org/10.1186/s43088-021-00153-4

[20] Saleh, A., Ibrahim, N. F., Hadhouda, K. M. (2022) Analytical and numerical water quality model for a sinusoidally varying pollutant discharge concentration. *Information Sciences Letters*, *11*(5), 1329–1334. https://doi.org/10.18576/isl/110501

[21] Ahsan, M. (2012). Numerical solution of the advection-diffusion equation using Laplace transform finite analytical method. *International Journal of River Basin Management*, *10*(2), 177–188. https://doi.org/10.1080/15715124.2012.679736

[22] Fazli, T. 2006. A Case Study of Sg. Skudai in the Comparison of River Water Quality Models in Total Maximum Daily Load Assessment. Master's thesis, Universiti Teknologi Malaysia

[23] Sattin, F., & Escande, D. F. (2025). Relation between diffusion equations and boundary conditions in bounded systems. *Foundations*, *5*(3), 26. https://doi.org/10.3390/foundations5030026

[24] Cekmer, O., Um, S., & Mench, M. M. (2025). A Novel Approach to Structure/Property Relations in Inhomogeneous Porous Media with High and Low Porosity. *Transport in Porous Media*, *152*(7). https://doi.org/10.1007/s11242-025-02180-6