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Application of Backward Differentiation Formula on Fourth-Order Differential Equations

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Abstract: Higher order ordinary differential equations are typically encountered in engineering, physical science, biological sciences, and numerous other fields. The analytical solution of the majority of engineering problems involving higher-order ordinary differential equations is not a simple task. Various numerical techniques have been proposed for higher-order initial value problems (IVP), but a higher degree of precision is still required. In this paper, we propose a novel two-step backward differentiation formula in the class of linear multistep schemes with a higher order of accuracy for solving ordinary differential equations of the fourth order. The proposed method was created by combining interpolation and collocation techniques with the use of power series as the basis function at some grid and off-grid locations to generate a hybrid continuous two-step technique. The method's fundamental properties, such as order, zero stability, error constant, consistency, and convergence, were explored, and the analysis showed that it is zero stable, consistent and convergent. The developed method is suitable for numerically integrating linear and nonlinear differential equations of the fourth order. Four Numerical tests are presented to demonstrate the efficiency and accuracy of the proposed scheme in comparison to some existing block methods. Based on what has been observed, the numerical results indicate that the proposed scheme is a superior method for estimating fourth-order problems than the method previously employed, confirming its convergence.

Keywords: Hybrid block scheme, self-starting, backward differentiation formula, fourth-order initial value problems, numerical estimation

1. Introduction

Numerical methods for solving higher order ordinary differential equations have gained serious attention from contemporary researchers due to their wide applications. Numerous physical problems encountered in virtually all areas of applied sciences and engineering are majorly transformed into differential equations, which sometimes takes the form:

$$y^{(iv)} = f\left(x, y, y', y'', y'''\right) \tag{1}$$

where f is a continuous real valued function containing the third, second and first derivatives combined with the following initial constraints

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$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y''(x_0) = y_0'', \quad y'''(x_0) = y_0''' \tag{2}$$

Analytical solutions to such physical problems rarely exist in some cases, therefore, this necessitates the need for suitable numerical techniques that will provide an estimated solution to higher ordinary differential equations. More so, initial value problems with respect to higher order Ordinary Differential Equations [ODE] play an essential role in describing various chemical, physical and biological phenomena such as beam theory, fluid dynamics, electric circuit, viscous fluid, quantum mechanics and neural networks, especially in the case of fourth order ODEs [1]. Authors in [2, 3] applied some numerical methods such as Homotopy continuation technique and Ostrowski Homotopy continuation technique for obtaining solutions to nonlinear ODEs. In recent times, numerical solution of higher order ODEs have been investigated through some kind of numerical techniques by several authors [4, 5, 6, 7, 8, 9]. Usually, the procedure for solving (1) is to reduce it into a system of first order initial value problems and use a suitable numerical technique for first order ODEs. Over the years, numerical techniques have been constructed to model solutions of (1), such techniques include; linear multistep technique, hybrid technique, block technique and Runge-Kutta technique [10, 11, 12, 13]. Recently, few researchers have made tremendous effort in developing block hybrid techniques for solutions to (1) directly, among them are [1, 14, 15, 16]. However, direct solutions of (1) through the approach of Backward differentiation formula technique have not been fully utilized, hence this motivates us to develop a hybrid block backward differentiation formula for computing fourth order ordinary differential equations utilizing power series as a basis function, to develop an implicit two-step hybrid block method with ten off grid points.

The main concept of this present work is concerned with the implementation of the proposed two-step linear multistep technique to directly provide estimated solutions for (1) in combination with the initial conditions in (2). The proposed scheme is designed as self-starting with high order of accuracy to provide an efficient scheme than some similar schemes available in literature. Some numerical properties in relation to the scheme will be examined for convergence and stability. It is expected that the scheme will handle fourth order problems efficiently. The introductory part of this work is discussed in section one, the second section concentrates on derivation of the proposed scheme and its convergence analysis are presented in the third section. We validate the convergence of the new scheme through some numerical experimentations in section four and conclusion of the research work is presented in section five.

2. Derivation of the Ten Points Block Scheme

In this section, we derive a numerical estimation to the fourth order ODE in (1) by making use of the power series as basis function

$$Y(x) = \sum_{j=0}^{d+c-1} g_j x^j \tag{3}$$

where d and c are numbers of interpolation and collocation points respectively, $x \in [x_0, x_N]$, and g_j 's are unknown coefficients to be determined. The fourth derivatives of (3) is given as:

$$Y'^{v}(x) = \sum_{j=0}^{d+c-1} j(j-1)(j-2)(j-3) g_j x^{j-4} \tag{4}$$

Hence equation (1) is equivalent to

$$f(x, y, y', y'', y''') = \sum_{j=0}^{d+c-1} j(j-1)(j-2)(j-3) g_j x^{j-4} \tag{5}$$

The unknown coefficients g_j 's are obtained from resolving a system of 13×13 system of non-linear algebraic equations in form of $KX = B$ which is generated from

$$\left.\begin{aligned} &Y\left(x_{m+j}\right)=y_{m+j} \\ &Y'\left(x_{m+j}\right)=y'_{m+j} \\ &Y''\left(x_{m+j}\right)=y''_{m+j} \\ &Y'''\left(x_{m+j}\right)=y'''_{m+j} \\ &j=0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},1,\frac{9}{8},\frac{5}{4},\frac{3}{2},\frac{7}{4},2 \end{aligned}\right\} \tag{6}$$

By applying inversion of matrix technique, values of $g_j's$ are obtained and inserted into (7) to obtain the continuous scheme of the Proposed method as:

$$\begin{aligned} y(x)=&\,\alpha_0 y_m+\alpha_{\frac{1}{8}}(x)y_{m+\frac{1}{8}}+\alpha_{\frac{1}{4}}(x)y_{m+\frac{1}{4}}+\alpha_{\frac{3}{8}}(x)y_{m+\frac{3}{8}} \\ &+\alpha_{\frac{1}{2}}(x)y_{m+\frac{1}{2}}+\alpha_{\frac{5}{8}}(x)y_{m+\frac{5}{8}}+\alpha_{\frac{3}{4}}(x)y_{m+\frac{3}{4}} \\ &\alpha_1 y_{m+1}+\alpha_{\frac{9}{8}}(x)y_{m+\frac{9}{8}}+\alpha_{\frac{5}{4}}(x)y_{m+\frac{5}{4}} \\ &+\alpha_{\frac{3}{2}}(x)y_{m+\frac{3}{2}}+\alpha_{\frac{7}{4}}(x)y_{m+\frac{7}{4}}+h^4\beta_2(x)f_{m+2} \end{aligned} \tag{7}$$

Evaluating (7) at $x=x_{m+2}$ and its derivatives at x_m, $x_{m+\frac{1}{8}}$, $x_{m+\frac{1}{4}}$, $x_{m+\frac{3}{8}}$, $x_{m+\frac{1}{2}}$, $x_{m+\frac{5}{8}}$, $x_{m+\frac{3}{4}}$, x_{m+1}, $x_{m+\frac{9}{8}}$, $x_{m+\frac{5}{4}}$, $x_{m+\frac{3}{2}}$, $x_{m+\frac{7}{4}}$ and x_{m+2} gives the sufficient number of schemes needed to implement as a block mode. The discrete schemes are presented below:

$$\begin{aligned} y_{m+\frac{1}{8}}=&\frac{134133238254457}{856508458873824}y_m+\frac{565215331712187}{190335213083072}y_{m+\frac{1}{4}}-\frac{145982907980936}{26765889339807}y_{m+\frac{3}{8}} \\ &+\frac{644891365705587}{95167606541536}y_{m+\frac{1}{2}}-\frac{16754048577942}{2973987704423}y_{m+\frac{5}{8}}+\frac{1518567171958291}{571005639249216}y_{m+\frac{3}{4}} \\ &-\frac{90347031135777}{95167606541536}y_{m+1}+\frac{17184249502381}{26765889339807}y_{m+\frac{9}{8}}-\frac{33409565983963}{190335213083072}y_{m+\frac{5}{4}} \\ &+\frac{7598606157181}{856508458873824}y_{m+\frac{3}{2}}+\frac{83564137665}{190335213083072}y_{m+\frac{7}{4}}-\frac{58779355089}{7613408523322880}h^4 f_{m+\frac{1}{8}} \\ &+\frac{413393799}{121814536373166080}h^4 f_{m+2} \end{aligned} \tag{8}$$

$$\begin{aligned} y_{m+\frac{1}{4}}=&\frac{5788822743533}{254917916202519}y_m+\frac{41289975054464}{368214767848083}y_{m+\frac{1}{8}}+\frac{686925346788352}{254917916202519}y_{m+\frac{3}{8}} \\ &-\frac{36811784310511}{9441404303797}y_{m+\frac{1}{2}}+\frac{96192670307584}{28324212911391}y_{m+\frac{5}{8}}-\frac{137156193972959}{84972638734173}y_{m+\frac{3}{4}} \\ &+\frac{5420020539249}{9441404303797}y_{m+1}-\frac{98702020052864}{254917916202519}y_{m+\frac{9}{8}}+\frac{2991247648879}{28324212911391}y_{m+\frac{5}{4}} \\ &-\frac{1355566160131}{254917916202519}y_{m+\frac{3}{2}}+\frac{96644416577}{368214767848083}y_{m+\frac{7}{4}}-\frac{22607444265}{1208499750886016}h^4 f_{m+\frac{1}{4}} \\ &-\frac{2438835}{1208499750886016}h^4 f_{m+2} \end{aligned} \tag{9}$$

$$
\begin{aligned}
y_{m+\frac{3}{8}} = {} & \frac{6192137042365}{130363511665408} y_m - \frac{39750732009657}{52960176614072} y_{m+\frac{1}{8}} + \frac{530231834575557}{260727023330816} y_{m+\frac{1}{4}} \\
& - \frac{59228118834435}{18623358809344} y_{m+\frac{1}{2}} + \frac{11270107942263}{2036929869772} y_{m+\frac{5}{8}} - \frac{123807766381647}{37246717618688} y_{m+\frac{3}{4}} \\
& + \frac{168291061865883}{130363511665408} y_{m+1} - \frac{3583097979449}{4073859739544} y_{m+\frac{9}{8}} + \frac{8976208791933}{37246717618688} y_{m+\frac{5}{4}} \\
& - \frac{1586444317691}{130363511665408} y_{m+\frac{3}{2}} + \frac{2035954589013}{3389451303300608} y_{m+\frac{7}{4}} + \frac{610400995155}{3873658632343552} h^4 f_{m+\frac{3}{8}} \\
& - \frac{285314805}{61978538117496832} h^4 f_{m+2}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
y_{m+\frac{1}{2}} = {} & -\frac{220235043499}{107124105827403} y_m + \frac{380821903936}{11902678425267} y_{m+\frac{1}{8}} - \frac{2000409990883}{7935118950178} y_{m+\frac{1}{4}} \\
& + \frac{11327398938112}{15303443689629} y_{m+\frac{3}{8}} + \frac{2523525952896}{3967559475089} y_{m+\frac{5}{8}} - \frac{1502703014413}{10202295793086} y_{m+\frac{3}{4}} \\
& - \frac{88131826367}{3967559475089} y_{m+1} + \frac{2379112310080}{107124105827403} y_{m+\frac{9}{8}} - \frac{24387712397}{3400765264362} y_{m+\frac{5}{4}} \\
& + \frac{44889575483}{107124105827403} y_{m+\frac{3}{2}} - \frac{175012177}{7935118950178} y_{m+\frac{7}{4}} + \frac{7535814755}{290198635892224} h^4 f_{m+\frac{1}{2}} \\
& + \frac{53245}{290198635892224} h^4 f_{m+2}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
y_{m+\frac{5}{8}} = {} & \frac{1622976625123}{27793538740224} y_m + \frac{7535814755}{3088170971136} y_{hz_m} - \frac{4959403425}{23958791714} y_{m+\frac{1}{8}} \\
& + \frac{5236275825}{10722815872} y_{m+\frac{1}{4}} - \frac{22180514300}{24880283703} y_{m+\frac{3}{8}} + \frac{197959481055}{171565053952} y_{m+\frac{1}{2}} \\
& + \frac{5886739075}{12063167856} y_{m+\frac{3}{4}} - \frac{62604246975}{343130107904} y_{m+1} + \frac{1702166485}{13571063838} y_{m+\frac{9}{8}} \\
& - \frac{372655647}{10722815872} y_{m+\frac{7}{4}} + \frac{7875}{10980163452928} h^4 f_{m+2}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
y_{m+\frac{5}{8}} = {} & \frac{54994653911}{9972916234449} y_m - \frac{2834493837696}{43215970349279} y_{m+\frac{1}{8}} + \frac{1145157901767}{3324305411483} y_{m+\frac{1}{4}} \\
& - \frac{1410008935424}{1424702319207} y_{m+\frac{3}{8}} + \frac{572589974673}{474900773069} y_{m+\frac{1}{2}} + \frac{18258612480}{3324305411483} y_{m+\frac{5}{8}} \\
& + \frac{3765028555971}{3324305411483} y_{m+1} - \frac{8331552577664}{107124105827403} y_{m+\frac{9}{8}} + \frac{97426899279}{474900773069} y_{m+\frac{5}{4}} \\
& - \frac{85799039461}{9972916234449} y_{m+\frac{3}{2}} + \frac{16293959631}{43215970349279} y_{m+\frac{7}{4}} + \frac{40693399677}{303936494764160} h^4 f_{m+\frac{3}{4}} \\
& - \frac{714177}{303936494764160} h^4 f_{m+2}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
y_{m+1} = {} & \frac{421783289898307}{3051615371125050} y_m - \frac{10550140657}{36167293287408} h^2 z_m - \frac{6533968224768}{8288338045031} y_{m+\frac{1}{8}} \\
& + \frac{1753151254374}{753485276821} y_{m+\frac{1}{4}} - \frac{3042397808472064}{671355381647511} y_{m+\frac{3}{8}} + \frac{91158592325997}{15069705536420} y_{m+\frac{1}{2}} \\
& + \frac{301662137660416}{56511395761575} y_{m+\frac{5}{8}} + \frac{53626548266606}{20344102474167} y_{m+\frac{3}{4}} + \frac{9164801709568}{13267892917935} y_{m+\frac{9}{8}} \\
& - \frac{3619033822614}{18837131920525} y_{m+\frac{5}{4}} + \frac{26759486744119}{2685421526590044} y_{m+\frac{3}{2}} - \frac{62432104946}{124325070675465} y_{m+\frac{7}{4}} \\
& + \frac{778155}{192892230866176} h^4 f_{m+2}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
y_{m+\frac{9}{8}} = {} & -\frac{6973395642881707}{59379459654977024} y_m - \frac{7121344943475}{475035677239816192} h^3 w_m + \frac{1259274372364485}{1507681592802151} y_{m+\frac{1}{8}} \\
& - \frac{41999242460503455}{14844864913744256} y_{m+\frac{1}{4}} + \frac{686691558024360}{115975507138627} y_{m+\frac{5}{8}} - \frac{487456851148298703}{59379459654977024} y_{m+\frac{1}{2}} \\
& + \frac{859828621379202}{115975507138627} y_{m+\frac{5}{8}} - \frac{55198422031433835}{14844864913744256} y_{m+\frac{3}{4}} + \frac{85408716742613055}{59379459654977024} y_{m+1} \\
& + \frac{4150240451370027}{14844864913744256} y_{m+\frac{5}{4}} - \frac{867638284254405}{59379459654977024} y_{m+\frac{3}{2}} + \frac{142855297539219}{192983243878675328} y_{m+\frac{7}{4}} \\
& - \frac{11424638925}{1900142708959264768} h^4 f_{m+2}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
y_{m+\frac{5}{4}} = {} & \frac{2170719929957}{283027224596445} y_m - \frac{37310500885376}{408817102194865} y_{m+\frac{1}{8}} + \frac{15401919775269}{31447469399605} y_{m+\frac{1}{4}} \\
& - \frac{62707864638464}{40432460656635} y_{m+\frac{3}{8}} + \frac{14197000545957}{4492495628515} y_{m+\frac{1}{2}} - \frac{130519677850368}{31447469399605} y_{m+\frac{5}{8}} \\
& + \frac{41051326203283}{13477486885545} y_{m+\frac{3}{4}} - \frac{81391837019253}{31447469399605} y_{m+1} + \frac{147579037948544}{56605444919289} y_{m+\frac{9}{8}} \\
& + \frac{19827789434429}{283027224596445} y_{m+\frac{3}{2}} + \frac{23682981573}{408817102194865} y_{m+\frac{7}{4}} - \frac{4521488853}{115007888089984} h^4 f_{m+\frac{5}{4}} \\
& - \frac{414087}{115007888089984} h^4 f_{m+2}
\end{aligned} \tag{16}
$$

$$\begin{aligned} y_{m+\frac{3}{2}} = & -\frac{9751907442461}{116947226934023} y_m + \frac{1482049188681408}{1520313950142299} y_{m+\frac{1}{8}} - \frac{1197272919089979}{31447469399605} y_{m+\frac{1}{4}} \\ & + \frac{1844976377570816}{116947226934023} y_{m+\frac{3}{8}} - \frac{3632262217046001}{116947226934023} y_{m+\frac{1}{2}} + \frac{4564096867813248}{116947226934023} y_{m+\frac{5}{8}} \\ & - \frac{6328520072711907}{233894453868046} y_{m+\frac{3}{4}} + \frac{2399836274145909}{116947226934023} y_{m+1} - \frac{2405077135150912}{116947226934023} y_{m+\frac{9}{8}} \\ & + \frac{1981734006254751}{233894453868046} y_{m+\frac{5}{4}} + \frac{342021577878963}{3040627900284598} y_{m+\frac{7}{4}} \\ & - \frac{4272806966085}{59876980190219776} h^4 f_{m+\frac{3}{2}} + \frac{2080637685}{59876980190219776} h^4 f_{m+2} \end{aligned} \tag{17}$$

$$\begin{aligned} y_{m+\frac{7}{4}} = & \frac{12557596427674831}{3322027173488787} y_m - \frac{15749883703635584}{369114130387643} y_{m+\frac{1}{8}} + \frac{79420163988442953}{369114130387643} y_{m+\frac{1}{4}} \\ & - \frac{2101139365763716096}{3322027173488787} y_{m+\frac{3}{8}} + \frac{434632554765946173}{369114130387643} y_{m+\frac{1}{2}} - \frac{510799466854972672}{369114130387643} y_{m+\frac{5}{8}} \\ & + \frac{979633998766418177}{1107342391162929} y_{m+\frac{3}{4}} - \frac{198288808654253031}{369114130387643} y_{m+1} + \frac{1541929236597726080}{3322027173488787} y_{m+\frac{9}{8}} \\ & - \frac{59424294401445337}{369114130387643} y_{m+\frac{5}{4}} + \frac{42960177351806923}{3322027173488787} y_{m+\frac{3}{2}} + \frac{6171832284345}{47246608689618304} h^4 f_{m+\frac{7}{4}} \\ & - \frac{628282954845}{47246608689618304} h^4 f_{m+2} \end{aligned} \tag{18}$$

$$\begin{aligned} y_{m+2} = & -\frac{41879859593}{1937780937} y_m + \frac{158449037312}{645926979} y_{m+\frac{1}{8}} - \frac{268060249600}{215308993} y_{m+\frac{1}{4}} \\ & + \frac{7146648535040}{1937780937} y_{m+\frac{3}{8}} + \frac{1492157190524}{215308993} y_{m+\frac{1}{2}} + \frac{5321603735552}{645926979} y_{m+\frac{5}{8}} \\ & - \frac{3451396556608}{645926979} y_{m+\frac{3}{4}} + \frac{729523112890}{215308993} y_{m+1} - \frac{5869299724288}{1937780937} y_{m+\frac{9}{8}} \\ & + \frac{712384376704}{645926979} y_{m+\frac{5}{4}} - \frac{205847630300}{1937780937} h^4 f_{m+\frac{3}{2}} + \frac{8481208768}{645926979} y_{m+\frac{7}{4}} \\ & + \frac{225225}{3444943888} h^4 f_{m+2} \end{aligned} \tag{19}$$

3. Convergence Properties of the Method

In this section, we examine the order and error constant for consistency and also zero stability. According [17], zero stability and consistency of a linear multistep method implies its convergence.

3.1 Order and Error Constant

Extending the analysis approach of [18] the local truncation error of fourth order equations can be written in the form:

$$L\left[y(x);h\right]=\sum_{j=0}^{k}\left(\alpha_j y_{n+j}-h\gamma_0 z_n-h^2\mu_0 Z_n-h^3\delta_0 w_n-h^4\left(\beta_j f_{n+j}\right)\right) \tag{20}$$

we can expand the terms in (20) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right]=c_0 y(x)+c_1 hy(x)+\ldots+c_p h^p y(x)+\ldots \tag{21}$$

where the constant c_p, $\mathrm{p}=0,1,\ldots$ are given as follows

$$\left.\begin{aligned}
c_0&=\sum_{j=0}^{k}\alpha_j\\
c_1&=\sum_{j=1}^{k}j\alpha_j\\
c_2&=\frac{1}{2!}\sum_{j=1}^{k}(j)^2\alpha_j\\
c_3&=\frac{1}{3!}\sum_{j=1}^{k}(j)^3\alpha_j\\
c_4&=\frac{1}{4!}\sum_{j=1}^{k}(j)^4\alpha_j-\sum_{j=0}^{k}\beta_j\\
&\vdots\\
c_p&=\frac{1}{\mathrm{p}!}\sum_{j=1}^{k}(j)^p\alpha_j-\frac{1}{(q-4)!}\left(\sum_{j=1}^{k}j^{p-4}\beta_j\right)
\end{aligned}\right\} \tag{22}$$

The numerical method is said to be of order p if $c_0=c_1=\ldots=c_p=0$ and $c_{p+4}\neq 0$. From the aforementioned analysis, the block method is of uniform order and the error constant is obtained and presented in the table below:

Table 1 - Order and error constants of the method

Equation	Order	Error constant
(13)	9	-6.2017×10^{-10}
(14)	9	2.5260×10^{-14}
(15)	9	5.6815×10^{-14}
(16)	9	-2.4552×10^{-15}
(17)	9	-9.7287×10^{-15}
(18)	9	-2.0830×10^{-14}
(19)	9	-5.5669×10^{-14}
(20)	9	8.5000×10^{-14}
(21)	9	3.2946×10^{-14}
(22)	9	-4.1699×10^{-13}
(23)	9	3.0376×10^{-11}
(24)	9	-1.6244×10^{-10}

And hence the method is consistent by virtue of order $p=7>1$.

3.2 Zero Stability

In order to characterize the method for stability, we rewrite the schemes (8) to (19) as a matrix difference equation as follows:

$$P^{(1)}Y_w = P^{(0)}Y_{w-1} + h\gamma_0 z_n + h^2\mu_0 Z_n + h^3\delta_0 w_n + h^4 Q^{(1)}F_w \qquad (23)$$

Where

$$\left.\begin{aligned}
Y_w &= \left(y_{n+\frac{1}{8}}, y_{n+\frac{1}{4}}, y_{n+\frac{3}{8}}, y_{n+\frac{1}{2}}, y_{n+\frac{5}{8}}, y_{n+\frac{3}{4}}, y_{n+1}, y_{n+\frac{9}{8}}, y_{n+\frac{5}{4}}, y_{n+\frac{3}{2}}, y_{n+\frac{7}{4}}, y_{n+2} \right)^T \\
Y_{w-1} &= \left(y_{n-\frac{1}{8}}, y_{n-\frac{1}{4}}, y_{n-\frac{3}{8}}, y_{n-\frac{1}{2}}, y_{n-\frac{5}{8}}, y_{n-\frac{3}{4}}, y_{n-1}, y_{n-\frac{9}{8}}, y_{n-\frac{5}{4}}, y_{n-\frac{3}{2}}, y_{n-\frac{7}{4}}, y_{n-2} \right)^T \\
F_{w+1} &= \left(f_{n+\frac{1}{8}}, f_{n+\frac{1}{4}}, f_{n+\frac{3}{8}}, f_{n+\frac{1}{2}}, f_{n+\frac{5}{8}}, f_{n+\frac{3}{4}}, f_{n+1}, {}_{n+\frac{9}{8}}, f_{n+\frac{5}{4}}, f_{n+\frac{3}{2}}, f_{n+\frac{7}{4}}, f_{n+2} \right)^T
\end{aligned}\right\}$$

and $P^{(1)}, P^{(0)}, Q^{(1)}$ are matrices whose entries are given by the coefficients of the block method.

Definition: A block linear multistep method is said to be zero stable if the roots (λ_k) of the difference equation in (23) as $h \to 0$ is $|\lambda_k| \le 1,\ k = 1,\dots 4$ and the multiplicity of the roots $|\lambda_k| = 1$ is not greater than the order of the ODE [17].

The first characteristics polynomial is given as: $\rho(\lambda) = \left|\lambda P^{(1)} - P^{(0)}\right| = \lambda^{12}(\lambda + 1)$. Hence our method is zero stable.

4. Numerical Experiments, Results and Discussion

Problem 1: We consider the following homogenous linear equation:

$$y^{(iv)} = 2y'''(t) - y''(t) + y(t);\quad y(0) = 1,\ y'(0) = -1,\ y''(0) = 0,\ y'''(0) = 1,\ 0 \le t \le 1$$

Exact solution:

$$y(t) = \left(\frac{1}{2} - \frac{1}{10}\sqrt{5}\right)e^{\frac{1}{2}(\sqrt{5}+1)t} + \left(\frac{1}{2} + \frac{1}{10}\sqrt{5}\right)e^{-\frac{1}{2}t(\sqrt{5}-1)} - \frac{2}{3}\sqrt{3}e^{\frac{1}{2}t}\sin\left(\frac{1}{2}\sqrt{3}t\right)$$

Table 2 - Comparison of the exact solution and the computed solution for problem 1 *(h*=0.1)

t	Exact Solution	Computed Results
0.1	0.9001795070863683729033744	0.9001795070863683729332272
0.2	0.8015444630147737801474346	0.8015444630147737804332314
0.3	0.7055988869753891027869434	0.7055988869753891034635962
0.4	0.6142389312914689286895104	0.6142389312914689289607135
0.5	0.5298081433937220262303676	0.5298081433937220237235878
0.6	0.4551605259035641675330235	0.4551605259035641574099734
0.7	0.3937326833438546937618270	0.3937326833438546679099793
0.8	0.349626585274874939064009	0.3496265852748748851633473
0.9	0.327704761421510586558073	0.3277047614215104870919633
1.0	0.333700082480454521066471	0.3337000824804543523292853

Table 3 - Comparison of absolute errors for problem 1 (*h*=0.1)

t	**[1]**	**Proposed method**
0.1	3.603×10^{-18}	2.985×10^{-20}
0.2	3.038×10^{-17}	2.858×10^{-19}
0.3	2.649×10^{-16}	6.767×10^{-19}
0.4	9.765×10^{-16}	2.712×10^{-19}
0.5	2.938×10^{-15}	2.507×10^{-18}
0.6	4.622×10^{-15}	1.012×10^{-17}
0.7	3.555×10^{-14}	2.585×10^{-17}
0.8	1.839×10^{-13}	5.390×10^{-17}
0.9	5.624×10^{-13}	9.947×10^{-17}
1.0	1.305×10^{-12}	1.687×10^{-17}

Problem 2. We consider the oscillatory problem:

$$y^{(iv)} = y(t) + 3\sin(2t+1);\quad y(0) = 1,\ y'(0) = 1 y''(0)1,\ y'''(0) = 0,\ 0 \le t \le 1$$

Exact solution:

$$y(t)=\frac{1}{5}\sin(2t+1)-\frac{1}{6}\sin(1)\cos t+\left(\frac{1}{10}\cos(1)+\frac{3}{4}+\frac{1}{20}\sin(1)\right)e^{t}+\left(\frac{1}{2}-\frac{1}{3}\cos(1)\right)\sin t$$
$$+\left(\frac{1}{4}+\frac{1}{20}\sin(1)-\frac{1}{10}\cos(1)\right)e^{-t}$$

Table 4 - Comparison of the exact solution and the computed solution for problem 2 (*h*=0.1)

t	Exact Solution	Computed Results
0.1	1.105007842811623187927536	1.105007842811623187951979
0.2	1.220128092730374411413894	1.220128092730374411629472
0.3	1.345661071542168229377346	1.345661071542168230060028
0.4	1.482127177330882548657100	1.482127177330882550258303
0.5	1.630280849447188408304729	1.630280849447188411613245
0.6	1.791121672193654017393253	1.791121672193654023430134
0.7	1.965902794734732817499420	1.965902794734732827557497
0.8	2.156137022961170458526985	2.156137022961170473856943
0.9	2.363601107916017626772622	2.363601107916017648602769
1.0	2.590338908970541769003346	2.590338908970541798128341

Table 5 - Comparison of absolute errors for problem 2 (*h*=0.1)

t	[1]	Proposed method
0.1	1.177×10^{-18}	2.444×10^{-20}
0.2	1.000×10^{-17}	2.156×10^{-19}
0.3	6.110×10^{-17}	6.827×10^{-19}
0.4	2.446×10^{-16}	1.601×10^{-18}
0.5	6.679×10^{-16}	3.309×10^{-18}
0.6	1.186×10^{-15}	6.037×10^{-18}
0.7	3.406×10^{-15}	1.006×10^{-17}
0.8	1.200×10^{-14}	1.533×10^{-17}
0.9	3.100×10^{-14}	2.183×10^{-17}
1.0	6.537×10^{-14}	2.912×10^{-17}

Problem 3: We also considered the system of linear fourth-order ODE.

$$y_1^{(iv)}=y_2(t)-3t+1;\quad y_1(0)=0,\ y_1'(0)=0, y_1''(0)=1,\ \ y_1'''(0)=2,\ 0\le t\le 1$$
$$y_2^{(iv)}=y_1(t)-17t;\quad y_2(0)=0,\ y_2'(0)=1, y_2''(0)=2,\ \ y_2'''(0)=3,\ 0\le t\le 1$$

Exact solution:

$$y_1(t)=17t-\frac{1}{2}\cos t-\frac{5}{4}e^{t}-6\sin t+\frac{9}{4}e^{-t}+\left(-\frac{1}{4}+\frac{7}{4}\sqrt{2}\right)e^{-\frac{1}{2}\sqrt{2}t}\cos\left(\frac{1}{2}\sqrt{2}t\right)+$$
$$\left(\frac{1}{4}-2\sqrt{2}\right)e^{-\frac{1}{2}\sqrt{2}t}\sin\left(\frac{1}{2}\sqrt{2}t\right)+\left(-\frac{1}{4}-\frac{7}{4}\sqrt{2}\right)e^{\frac{1}{2}t\sqrt{2}}\cos\left(\frac{1}{2}\sqrt{2}t\right)+\left(-\frac{1}{4}-2\sqrt{2}\right)e^{\frac{1}{2}\sqrt{2}t}\sin\left(\frac{1}{2}\sqrt{2}t\right)$$

$$y_2(t)=3t-1-\frac{1}{2}\cos t-\frac{5}{4}e^{t}-6\sin t+\frac{9}{4}e^{-t}-\left(-\frac{1}{4}+\frac{7}{4}\sqrt{2}\right)e^{-\frac{1}{2}\sqrt{2}t}\cos\left(\frac{1}{2}\sqrt{2}t\right)-$$
$$\left(\frac{1}{4}-2\sqrt{2}\right)e^{-\frac{1}{2}\sqrt{2}t}\sin\left(\frac{1}{2}\sqrt{2}t\right)-\left(-\frac{1}{4}-\frac{7}{4}\sqrt{2}\right)e^{\frac{1}{2}t\sqrt{2}}\cos\left(\frac{1}{2}\sqrt{2}t\right)-\left(-\frac{1}{4}-2\sqrt{2}\right)e^{\frac{1}{2}\sqrt{2}t}\sin\left(\frac{1}{2}\sqrt{2}t\right)$$

Table 6 - Comparison of the exact solution and the computed solution for problem 3 (*h*=0.1)

t	**Exact Solution** $y_1(t)$	**Computed Results** $y_1(t_n)$
0.1	0.0053373361705881012526558	0.005337336170588101252487778
0.2	0.0227281853728687592939247	0.02272818537286875929285773
0.3	0.0542991542581902639454504	0.05429915425819026394237055
0.4	0.102241674099460735843777	0.1022416740994607358378167
0.5	0.168797961870536684333329	0.1687979618705366843291029
0.6	0.256249792595063460097506	0.2562497925950634601070174
0.7	0.366910274865422174252677	0.3669102748654221742980689
0.8	0.50311875451691018519534	0.5031187545169101853094820
0.9	0.667238888199921522642550	0.6672388881999215228176294
1.0	0.861659829232760281676616	0.8616598292327602817759447

Table 7 - Comparison of the exact solution and the computed solution for problem 3 (*h*=0.1)

x	**Exact Solution** $y_2(x)$	**Computed Results** $y_2(x_n)$
0.1	0.1104985847621473221798622	0.1104985847621473221780658
0.2	0.2439547606956488032809573	0.2439547606956488032687760
0.3	0.4031568508078514524899636	0.4031568508078514524520482
0.4	0.590555687251049340149771	0.5905556872510493400405476
0.5	0.808097804945184635366405	0.8080978049451846351158607
0.6	1.057060273201680465889844	1.057060273201680465383110
0.7	1.337887389382948267530319	1.337887389382948266628800
0.8	1.650029462637979404344419	1.650029462637979402870930
0.9	1.991783918939935203311369	1.991783918939935201092457
1.0	2.360138961328219279871556	2.360138961328219276693779

Table 8 - Comparison of absolute errors for problem 3 (*h*=0.1)

x	[1]		Proposed method	
	$\lvert y_1(x)-y_1(x_n)\rvert$	$\lvert y_2(x)-y_2(x_n)\rvert$	$\lvert y_1(x)-y_1(x_n)\rvert$	$\lvert y_2(x)-y_2(x_n)\rvert$
0.1	1.177×10^{-16}	8.839×10^{-10}	1.690×10^{-22}	1.796×10^{-21}
0.2	5.00×10^{-14}	5.775×10^{-09}	1.073×10^{-21}	1.218×10^{-20}
0.3	1.065×10^{-13}	9.557×10^{-08}	4.285×10^{-21}	3.792×10^{-20}
0.4	1.732×10^{-13}	2.493×10^{-07}	1.566×10^{-20}	1.092×10^{-19}
0.5	1.280×10^{-11}	8.721×10^{-07}	4.781×10^{-20}	2.507×10^{-19}
0.6	8.137×10^{-11}	1.847×10^{-06}	1.324×10^{-19}	5.077×10^{-19}
0.7	3.901×10^{-10}	1.761×10^{-04}	3.205×10^{-19}	9.054×10^{-19}
0.8	4.131×10^{-09}	1.046×10^{-03}	6.913×10^{-19}	1.484×10^{-18}
0.9	4.371×10^{-08}	3.187×10^{-03}	1.323×10^{-18}	2.240×10^{-18}
1.0	2.774×10^{-07}	3.187×10^{-03}	2.345×10^{-18}	3.207×10^{-18}

Problem 4: We also consider the nonlinear fourth-order ODE initial value problem: [10].

$$y^{(iv)} = (y')^2 - yy''' - 4x^2 + e^x(1-4x+x^2); \quad y(0)=1, \ y'(0)=1, y''(0)=3, \ y'''(0)=1$$

Exact solution:

$$y(x) = x^2 + e^x$$

Table 9 - Comparison of the exact solution and the computed solution for problem 4

x	Exact Solution $y(x)$	Computed Results $y(x_n)$
0.031250	1.032719969999102670938748	1.032719969999102670975280
0.062500	1.068400708917859429563391	1.068400708917859429846103
0.093750	1.107074202807825848650210	1.107074202807825849655353
0.125000	1.148773453066826316829007	1.148773453066826320056659
0.156250	1.193532508669504402298185	1.193532508669504410658578
0.187500	1.241386499420980710655586	1.241386499420980729237603
0.218750	1.292371670266095154946199	1.292371670266095191390574
0.250000	1.346525416687741484073421	1.346525416687741549360843
0.281250	1.403886321228865568935238	1.403886321228865677743025
0.312500	1.464494191173796362838757	1.464494191173796534346352

Table 10 - Comparison of absolute errors for problem 4

x	[19]	[10]	Proposed method
0.031250	1.149×10^{-12}	1.788×10^{-10}	3.653×10^{-20}
0.062500	1.885×10^{-11}	1.134×10^{-08}	2.827×10^{-19}
0.093750	9.780×10^{-11}	1.196×10^{-07}	1.005×10^{-18}
0.125000	3.166×10^{-10}	6.401×10^{-07}	3.228×10^{-18}
0.156250	7.909×10^{-10}	2.349×10^{-06}	8.360×10^{-18}
0.187500	1.676×10^{-09}	6.573×10^{-06}	1.858×10^{-17}
0.218750	3.169×10^{-09}	1.610×10^{-05}	3.644×10^{-17}
0.250000	5.512×10^{-09}	3.501×10^{-05}	6.529×10^{-17}
0.281250	8.995×10^{-09}	6.985×10^{-05}	1.088×10^{-16}
0.312500	1.396×10^{-08}	1.245×10^{-04}	1.715×10^{-16}

4.1 Discussion of Results

Problem 1 is a homogeneous linear equation solved by the proposed method, and its exact and numerical solutions are listed in Table 2. The comparison in Table 3 between the proposed method and that of [1] demonstrates that the proposed method is more accurate. Table 4 displays the oscillatory problem's exact and numerical solutions, which differ by a negligible amount. Table 5 demonstrates that the absolute errors of the proposed method are less than those of [1]. In addition, we considered the system of fourth-order equations in problem 3, whose exact and numerical solutions are presented in tables 6 and 7. Table 8 displays the corresponding absolute errors relative to [1]. The Proposed method is then used to solve the nonlinear ODE of fourth order in problem 4. The Proposed method is better than those of [10], [19] as shown by a comparison of the absolute errors.

5. Conclusion

This paper considers the derivation and implementation of a class of linear multistep method in providing an estimate solution to general fourth order ordinary differential equations involving a set of initial conditions. The paper specifically considers a two-step backward differentiation formula with some off-grid points in the derivation process. Direct implementation of the derived method on general fourth order initial value problems is possible (without necessarily need to reduce to system of first order equations) due to the self-starting nature of the method. The point collocation technique adopted in deriving the method enables us to get the continuous form of the method through which all the necessary schemes required to implement the method directly on fourth order equations are generated to form the block method. The result of comparing the estimated solutions with the exact solutions exhibited very negligible errors, thus clearly confirming that the method is not only convergent, but very effective in handling a general fourth order initial value problems. Further validation of the suitability of our method is carried out by comparing the absolute errors with the recent methods in the literature as cited therein, which shows that our method is superior in terms of accuracy than those methods. In future study, we plan to investigate the application of the proposed method to partial integro-differential equations and fractional differential problems.

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