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JST

http://penerbit.uthm.edu.my/ojs/index.php/jst
ISSN : 2229-8460 e-ISSN : 2600-7924

Journal of Science and Technology

Introduction of Ostrowski Homotopy Continuation Method for Solving Nonlinear Equations Using Mathematica

Hafizudin Mohamad Nor[1*], Asma Izzati Asnor[2], Zainor Ridzuan Yahya[1], Muhammad Zaini Ahmad[1]

[1]Faculty of Applied and Human Sciences,
Universiti Malaysia Perlis, 02600 Arau, Perlis, MALAYSIA

[2]School of Distance Education,
Universiti Sains Malaysia, 11800 Gelugor, Pulau Pinang, MALAYSIA

*Corresponding Author

DOI: https://doi.org/10.30880/jst.2022.14.01.004
Received 10 November 2021; Accepted 19 May 2022; Available online 28 June 2022

Abstract: The solution to a nonlinear equation is found in this study by combining a classical and a powerful method. Basically, it is well known that the Homotopy Continuation Method (HCM) is a powerful method that has been used for solving the problem of the classical method. A new approach is introduced in this study which is known as the Ostrowski Homotopy Continuation Method (Ostrowski-HCM) with a purpose to overcome the divergence problem that arises from the classical Ostrowski's method when a bad initial guess is used. To put it simply, when the derivative of a given function at the starting point is equal to zero, the problem arises. As a result, the division by zero renders the scheme invalid. In addition, a mathematical software, *Mathematica 7.0*, is used to implement the Ostrowski-HCM results. Thus, from the analysis of the results, it is proven that the Ostrowski-HCM is reliable and advantageous for solving the nonlinear equation.

Keywords: Numerical method, **n**onlinear equations, Ostrowski's method, Homotopy Continuation Method

1. Introduction

One of the oldest known approximation problems is the root-finding problem. To date, research in this area is still ongoing in order to solve the nonlinear equations. Fundamentally, the nonlinear equations are divided into several categories, including nonlinear algebraic, exponential, logarithmic, and trigonometric equations. The nonlinear equations in its general form,

$$f(x) = 0, \tag{1}$$

where $x \in \Re$.

There are a number of conventional methods for solving the nonlinear equations, including Newton's method, bisection method, secant method, Ostrowski's method, and others. For example, there are numerous studies regarding finding the solution of nonlinear equations using Newton's method and its variants. In (Silalahi, Laila & Sitanggang, 2017) [1] discussed the performance of three methods i.e., Newton's method, Halley's method and their proposed method that is Newton Inverse Halley method. (Villafuerte et al., 2019) [2] introduced an iterative Newton-type method to solve nonlinear equations and related to the application in electrical power systems. Furthermore (Salas, 2021) [3] also proposed a new method known as Halley's method Rediscovered for solving nonlinear and transcendental equations.

*Corresponding author: hafizudin@unimap.edu.my
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Alexander Markowich Ostrowski (Ostrowski, 1960) is the pioneer of the Ostrowski's method. The method introduced is aimed to find the roots of a single-variable nonlinear function by extending Ostrowski's method [4]. Subsequently, Ostrowski's method has been presented in detail by the same pioneer [5]. This method was formulated to have locally quintic convergent, which doubles the performance of Newton's method. The performance of Ostrowski's method has been affected since there is a divergence problem with Newton's method. Therefore, we have to determine the given function's derivative. The problem can occur if it is more difficult to derive than what is already given.

The Ostrowski method does not work in the first stage when $f'(x_0) = 0$ or $f'(x_0) \approx 0$. This issue will result in a slow convergence or divergence problem. The homotopy continuation method (HCM) provides a useful approach for determining the zeros of each function in terms of convergent ways in finding the approximate solutions as an initiative to tackle this problem. With the easier calculation zeros function, the homotopy method transforms a difficult or complicated problem into a simpler one [6].

2. Methodology

To solve algebraic and transcendental equations, we will use the Ostrowski-HCM method. Hence, we select the standard homotopy and auxiliary homotopy functions for a better understanding of the basic concept of the chosen method.

2.1 Homotopy Function

Homotopy refers to the homotopy function, $H(x,t)$ and continuous refers to the nonlinear equation being embedded in a one-parameter family, t, of problems which run over the interval $t \in [0,1]$ [7].

$$H(x,t) = (1-t)g(x) + tf(x). \tag{2}$$

Since our target is to solve $H(x,t) = 0$, thus we have the following conditions:

$$H(x,0) = g(x) = 0, \qquad \text{(Initial start system)} \tag{3}$$

$$H(x,1) = f(x) = 0. \qquad \text{(Target solution)} \tag{4}$$

2.2 Auxiliary Homotopy Function

In addition, the auxiliary homotopy function is another important function. Basically, the auxiliary homotopy function $g(x)$ can be identified using several ways, for instance, Newton, fixed-point, and affine functions. For this study, the fixed-point function is selected as our auxiliary homotopy function since the function is simpler to use rather than others. In general, the fixed-point function is expressed as

$$g(x) = x - x_0, \tag{5}$$

where x_0 refers to the initial guess. The most significant aspect of the auxiliary homotopy function is that this function must be controllable and uncomplicated to solve.

2.3 Ostrowski Homotopy Continuation Method

Besides, as discussed in [8,9,10], there are several types of HCMs, including Newton-HCM, secant-HCM, and Adomian-HCM. Therefore, in this study, we attempt to solve the nonlinear equations by using the Ostrowski-HCM with the combination of the classical and powerful methods.

In this section, we will show how it turned out into an idea of introducing the aforementioned formula of the Ostrowski-HCM. Firstly, a classical method known as Ostrowski's method is chosen before we do the combination. The Ostrowski's method employs two-step iterations using the following formula,

$$y_i = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad i = 0,1,2,...,k. \tag{6a}$$

$$x_{i+1} = y_i - \frac{x_i - y_i}{f(x_i) - 2f(y_i)} f(y_i). \qquad i = 0,1,2,...,k. \tag{6b}$$

where (6a) represents the classical Newton's method whereas (6b) represents Ostrowski's formula. Then, the pioneer of the method [5] combine (6a) and (6b), subsequently, the formula becomes,

$$y_i = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad x_{i+1} = y_i - \frac{f(x_i)}{f(x_i)-2f(y_i)}\frac{f(y_i)}{f'(x_i)}. \qquad i = 0,1,2,...,k. \tag{7}$$

The Ostrowski's method (7), as previously stated, does not perform at the initial stage when $f'(x_0) = 0$ or $f(x_0) \approx 0$. To overcome this divergence problem, (Nor et al., 2014) [11] have developed the following Ostrowski-HCM,

$$y_i = x_i - \frac{H(x_i,t)}{H'(x_i,t)}, \qquad x_{i+1} = y_i - \frac{H(x_i,t)}{H(x_i,t)-2H(y_i,t)}\frac{H(y_i,t)}{H'(x_i,t)}. \qquad i = 0,1,2,...,k. \text{ and } t \in [0,1]. \tag{8}$$

In addition, a technique described by (Palancz et al., 2010) [12] is used to improve the accuracy of approximation solutions in this work,

$$x_{i+1} = \text{Ostrowski}(H(x,t_{i+1}),\{x,x_i\}) \tag{9}$$

where x_i is the initial value for calculating next x_{i+1}. Note that, each t_{i+1} is iterated two times only.

3. Algorithm

Step 1: Given the initial function $f(x), g(x), H(x,t)$ and x_0.
Step 2: Given the formula of Ostrowski-HCM.
Step 3: Assign $x_0 = a, t = 0$ and $k = 1$.
Step 4: For $Abs(f(x)) \geq 10^{-6}$

$k = k+1$

For $i = 1, i \leq k$

$t = t + \frac{1}{k}$

$x = f(x,t).$

Step 5: The value of x is then revised to the better approximation for every $H(x,t) = 0$.
Step 6: When $t = 1$, the solution of $f(x) = 0$ is found.

4. Numerical Example and Discussion

The numerical experiments were analyzed to evaluate the method's performance. The proposed method was applied to several examples of scalar nonlinear equations. Note that the stopping criterion utilized is $|f(\tilde{x}_k)| < 10^{-6}$. Therefore, the equations tested are as follows

(i) $f(x) = x^2 + 8x - 9 = 0.$ [12];

(ii) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 2 = 0.$ [9];

(iii) $f(x) = x^4 + 6x - 40 = 0.$ [13];

(iv) $f(x) = x^5 + x^4 - x^2 - 7x + 4 = 0.$ [14];

(v) $f(x) = x - \cos x = 0.$ [15];

(vi) $f(x) = \sin x - \frac{\cosh x}{1000} + 0.5 = 0.$ [16];

(vii) $f(x) = e^{-x} + \cos x = 0.$ [17].

The following is the pseudo-code for the equation (i) in Mathematica 7.0 by using Ostrowski's method as well as Ostrowski-HCM.

```
OstrowskiMethod[x0_] := Module[{a = x0},
  StartTime = SessionTime[];
  Clear[f, g, H, x, t, i];
  f[x_] = x^2 + 8 x - 9;
  y = x - f[x] / D[f[x], x];
  K[x_] = y - f[x]/(f[x] - 2 f[y]) f[y]/D[f[x], x];
  x = a;
  Print[" x0=", x];
  For[i = 1, Abs[f[x]] ≥ 10^-6, i++,
   x = K[x];
   Print[" x", i, " = ", SetPrecision[x, 15]];];
  If[f'[a] == 0, Print["Number of iterations required to achieve the stopping criterion f(x) < 10^-6 = ", Indeterminate],
   Print["Number of iterations required to achieve the stopping criterion f(x) < 10^-6 = ", i - 1]];
  EndTime = SessionTime[];
  DifferenceTime = EndTime - StartTime;
  Print["Time Used is = ", DifferenceTime, "   second"];
  Print[" x = ", SetPrecision[x, 15], " and  f[x] =  ", f[x]]]
```

```
OstrowskiHCM[x0_] := Module[{a = x0},
  StartTime = SessionTime[];
  Clear[f, g, H, x, i, t, k, y];
  f[x_] = x^2 + 8 x - 9;
  g[x_] = (x - a);
  H[x_, t_] = (1 - t) (g[x]) + t (f[x]);
  y = x - H[x, t] / D[H[x, t], x];
  f[x_, t_] = y - H[x, t]/(H[x, t] - 2 H[y, t]) H[y, t]/D[H[x, t], x];
  x = a; t = 0;
  Print["t=0  ,      x0=", x]; k = 1;
  For[w = 1, Abs[f[x]] ≥ 10^-6, w++,
   k = k + 1; t = 0; x = a;
   Print["------------------------If number of iterations = ", k, "-------------"];

   For[i = 1, i ≤ k, i++,
    Print["t=", t = t + 1/k, ",      x", i, " = ", x = SetPrecision[f[x, t], 15]];
    For[j = 1, j ≤ 1, j++,
     x = SetPrecision[f[x, t], 15];
     Print["                ", x];];]];
  EndTime = SessionTime[];
  DifferenceTime = EndTime - StartTime;
  Print["Time Used is = ", DifferenceTime, "   second"];
  Print["Number of iterations required to achieve the stopping criterion f(x) < 10^-6 = ", k];
  Print[" x = ", SetPrecision[x, 15], " and  f[x] =  ", f[x]];]
```

The result obtained is as follows

```
OstrowskiMethod[-4.]
x0=-4.
Power::infy : Infinite expression 1/0. encountered. >>
Power::infy : Infinite expression 1/0. encountered. >>
Power::infy : Infinite expression 1/0. encountered. >>
General::stop : Further output of Power::infy will be suppressed during this calculation. >>
∞::indet : Indeterminate expression -9 + ComplexInfinity + ComplexInfinity encountered. >>
∞::indet : Indeterminate expression -9 + ComplexInfinity + ComplexInfinity encountered. >>
x1 = Indeterminate
Number of iterations required to achieve the stopping criterion f(x) < 10^-6 = Indeterminate
Time Used is = 0.5928010   second
x = Indeterminate and  f[x] =  Indeterminate
```

As we see the data, we can see there was a divergence issue for the Ostrowski's method when started at $x_0 = -4$ (bad initial guess). Due to the division by zero, in which $f'(x_0) = 0$, thus, it is clear that the method unable to perform at the initial stage. Hence, the aforementioned problems were solved by applying Ostrowski-HCM. The output is as follows

```
OstrowskiHCM[-4.]
t=0  ,      x0=-4.
------------------------If number of iterations = 2-------------
t=1/2,     x1 = 8.74509803921569
              0.954371583319900
t=1,      x2 = 1.00000000441455
              1.00000000000000
Time Used is = 0.0468000   second
Number of iterations required to achieve the stopping criterion f(x) < 10^-6 = 2
x = 1.00000000000000 and  f[x] =  0.×10^-14
```

Then, we also take another initial point, $x_0 = -3$ for solving the similar Equation (i). The following Table 1 presents the results obtained.

Table 1 - Performance of Ostrowski's method and Ostrowski-HCM

Ostrowski's method	Ostrowski-HCM
$x_0 = -3$	$x_0 = -3$
$x_1 = 3.46153846153846$	$x_1 = 0.624134940718965$
$x_2 = 1.01524760194490$	$x_2 = 1$
$x_3 = 1.00000000005372$	

Graphically, the results can be illustrated as follows

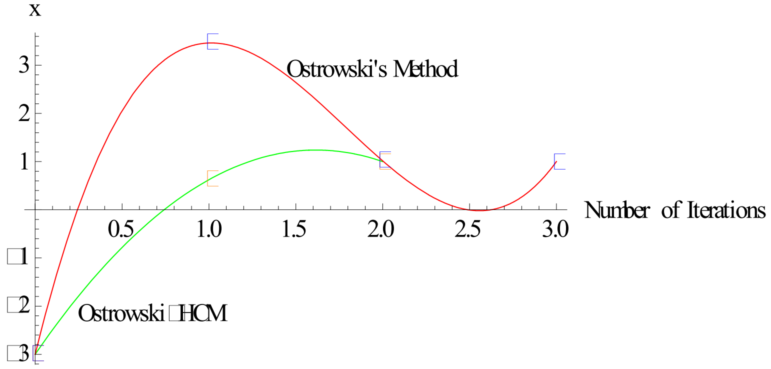

Fig. 1 - Performance of Ostrowski's method and Ostrowski-HCM in terms of number of iterations needed

Table 1 and fig. 1 presented previously to demonstrate the performance of Ostrowski's method and the Ostrowski-HCM when both methods were performed at initial guess $x_0 = -3$. Unlike Ostrowski's method, which required three iterations, the Ostrowski-HCM just required two iterations to converge to the actual root $x = 1$. Besides, as described in (Palancz et al., 2010) [12], another actual root $x = -9$ can be found with different initial values. Thus, the advantage of the proposed method over its classical method can be shown from the two initial guesses selected. However, making a thorough conclusion with only one example, on the other hand, is insufficient. Therefore, several equations were chosen, and the resulting findings are displayed in Table 2.

Table 2 - Comparison between Ostrowski-HCM and Ostrowski method

Equation	Initial Value x_0	Ostrowski's method	Ostrowki-HCM
(i)	–4	Diverge	2
	–3	3	2
(ii)	–2	Diverge	2
	3	Diverge	2
	3.01	7	2
(iii)	–1	5	3
	0	4	2
(iv)	1	Diverge	3
	–10	6	4
(v)	–2	10	3
	0	2	2
	2	2	2
(vi)	0	2	2
	–4	3	3
(vii)	–3	3	2
	4	2	2

After being analyzed, a similar behavior occurs when the numerical experiment is conducted on other equations with different initial values. This can be demonstrated by comparing the numerical results obtained by the two methods. The Ostrowski-HCM manage to solve the nonlinear equations with a fewer number of iterations as opposed to the Ostrowski's method which encounters a divergence problem. The Ostrowski-HCM has the advantage of being able to tackle the divergence problem at the point of a bad initial guess, as (Wu, 2005) [8] demonstrated.

5. Conclusion

Based on the analysis of results, the Ostrowski-HCM can overcome the drawback of its classical method. The number of iterations required is determined by the given stopping condition, and the optimal method is determined by the approach that required the fewest iterations. All in all, the benefits of the Ostrowski-HCM are its capability to handle the divergence problem for the nonlinear equations and the reduced number of iterations required. As a result, the Ostrowski-HCM can be recommended as a method for solving the nonlinear equations.

Acknowledgements

The corresponding author wishes to express his gratitude to the Universiti Malaysia Perlis, especially to the Institute of Engineering Mathematics, Faculty of Applied and Human Sciences for preparing the comfortable work conditions to write this paper.

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